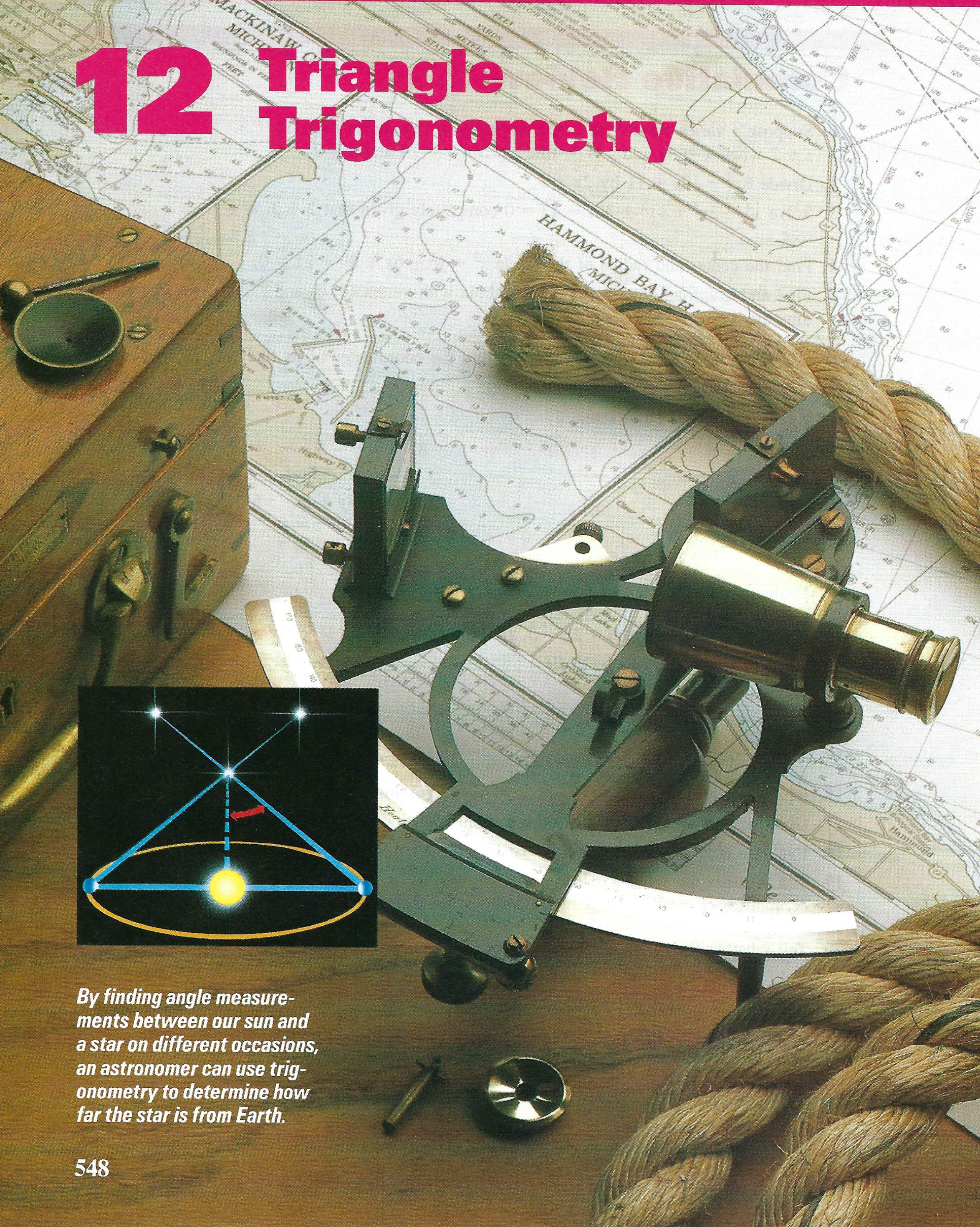
12 Triangle Trigonometry



By finding angle measurements between our sun and a star on different occasions, an astronomer can use trigonometry to determine how far the star is from Earth.

Trigonometric Functions

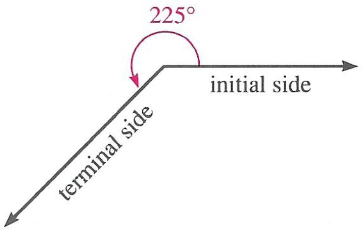
12-1 Angles and Degree Measure

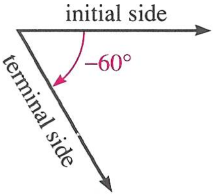
Objective To use degrees to measure angles.

Trigonometry, which means ''triangle measurement,'' is a branch of mathematics that was originally used for surveying. Today trigonometry is used in astronomy, navigation, architecture, and many other fields where measurement is important.

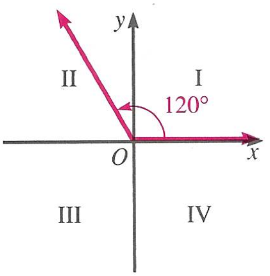
In this chapter angle measurements will be given in *degrees*. Recall that one **degree** (1°) is $\frac{1}{360}$ of a complete revolution. In geometry most angles you work with have measures between 0° and 180°. In trigonometry you will also use angles with measures greater than 180° and less than 0°.

You can generate any angle by using a rotation that moves one of the rays forming the angle, called the **initial side,** onto the other ray, called the **terminal side.** As shown in the figure below, *counterclockwise* rotations produce **positive angles** and *clockwise* rotations produce **negative angles.** An angle generated in this manner is called a **directed angle.**

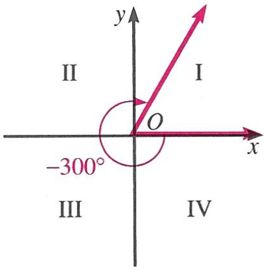




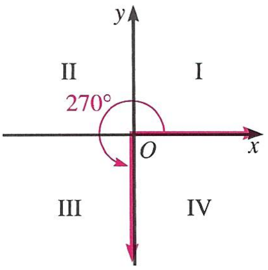
An angle is in **standard position** when its initial side coincides with the positive x-axis. As shown in the diagrams below, angles in standard position can be classified according to the quadrant in which their terminal side lies. If the terminal side lies on a coordinate axis, the angle is called a **quadrantal angle.**



Second-quadrant angle

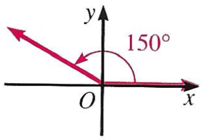


First-quadrant angle

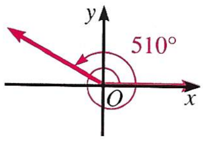


Quadrantal angle

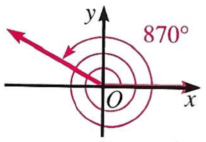
Two angles in standard position are **coterminal** if their terminal sides coincide. The diagrams below show several angles coterminal with a 150° angle.



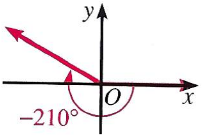
$150° + 0 \bullet 360°$



$150° + 1 \bullet 360°$



$150° + 2 \bullet 360°$



$150° + (-1) \bullet 360°$

The measures of all angles coterminal with a 150° angle are given by the expression

$$150° + n \cdot 360°, \text{ where } n \text{ is an integer.}$$

In the diagrams above, n has the values 0, 1, 2, and -1, respectively.

Example 1 **a.** Write a formula for the measures of all angles coterminal with a 30° angle.

b. Use the formula to find two positive angles and two negative angles that are coterminal with a 30° angle.

Solution **a.** $30° + n \cdot 360°$, where n is an integer.

b. To find the measures of the angles, let n equal 1, 2, -1, and -2.

$$30° + (1)360° = 390°$$
$$30° + (2)360° = 750°$$
$$30° + (-1)360° = -330°$$
$$30° + (-2)360° = -690°$$

To measure angles more precisely than to the nearest degree, you can use either decimal degrees or degrees, minutes, and seconds. One **minute** (1′) is $\frac{1}{60}$ of 1°, and one **second** (1″) is $\frac{1}{60}$ of 1′.

Many calculators will convert decimal degrees to degrees, minutes, and seconds, and vice versa. (See the Calculator Key-In on page 554.) If you do not use a calculator, you can use the methods shown in Example 2.

Example 2 **a.** Express 14°36′54″ in decimal degrees.

b. Express 72.568° in degrees, minutes, and seconds to the nearest second.

Solution **a.** Use these facts: $1' = \left(\frac{1}{60}\right)^{\circ}$ and $1'' = \left(\frac{1}{60}\right)' = \left(\frac{1}{3600}\right)^{\circ}$.

$$14°36'54'' = 14° + \left(\frac{36}{60}\right)^{\circ} + \left(\frac{54}{3600}\right)^{\circ}$$
$$= 14° + 0.6° + 0.015°$$
$$= 14.615° \quad \textbf{\textit{Answer}}$$

b.
$$\begin{aligned} 72.568^\circ &= 72^\circ + (0.568 \times 60)' \\ &= 72^\circ + 34.08' \\ &= 72^\circ + 34' + (0.08 \times 60)'' \\ &= 72^\circ + 34' + 4.8'' \\ &\approx 72^\circ 34' 5'' \end{aligned}$$

Answer

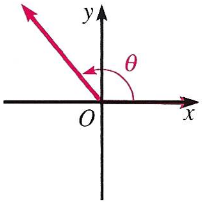
We often use capital letters, as in $\angle A$, or Greek letters such as θ (theta), ϕ (phi), and α (alpha) to name angles. (A list of Greek letters used in this book appears on page xvi.) To denote the measure of an angle, we may write, for example, $\theta = 20^\circ$, $\angle A = 20^\circ$, $m\angle A = 20^\circ$, or $\angle A = 20$. In this book we will use $\theta = 20^\circ$ or $\angle A = 20^\circ$ to denote an angle with a measure of 20°. Also, when we say that 120° is a second-quadrant angle, we mean that a 120° *angle* in standard position has its terminal side in the second quadrant.

Oral Exercises

1. The measure of an angle in quadrant I is between 0° and 90°. What can you say about the measure of an angle in quadrant II? in quadrant III? in quadrant IV?
2. Give the measures of the quadrantal angles from 0° to 360°, inclusive.

Estimate the measure of angle θ.

3.



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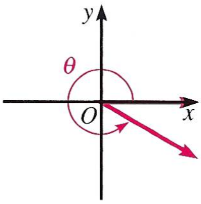

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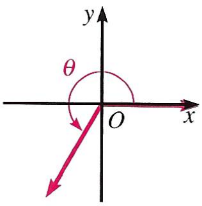

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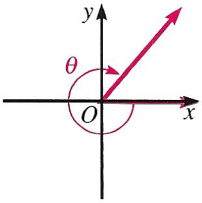

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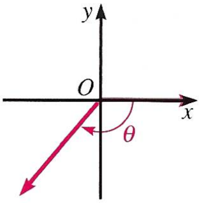

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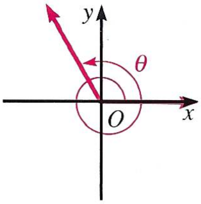

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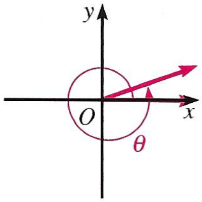

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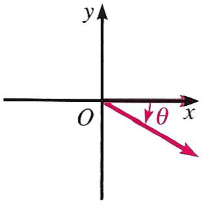


Name the quadrant of each angle.

11. 140° **12.** 215° **13.** −45° **14.** −150°

15. −315° **16.** 440° **17.** −400° **18.** 500°

Give two angles, one positive and one negative, that are coterminal with the given angle.

19. 50° **20.** 100° **21.** 120° **22.** 40°

23. −45° **24.** −210° **25.** 390° **26.** 480°

Written Exercises

Sketch each angle in standard position. Indicate its rotation by a curved arrow. Classify each angle by its quadrant. If the angle is a quadrantal angle, say so. (See the diagrams at the bottom of page 549.)

A **1. a.** 135° **b.** −135° **2. a.** 40° **b.** −40°

3. a. 300° **b.** −300° **4. a.** 240° **b.** −240°

5. −270° **6.** 315° **7.** 290° **8.** −90°

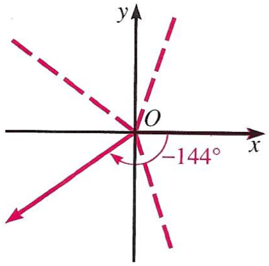
9. 495° **10.** 1080° **11.** −810° **12.** 750°

Sketch in standard position the angle described and then find its measure.

Sample 1 $\frac{2}{5}$ of a clockwise rotation

Solution $\frac{2}{5} \times (-360) = -144$

$\therefore$ the measure is −144°.



13. $\frac{2}{3}$ of a counterclockwise revolution

14. $\frac{3}{8}$ of a counterclockwise revolution

15. $\frac{3}{4}$ of a clockwise revolution **16.** $\frac{1}{6}$ of a clockwise revolution

17. $1\frac{3}{5}$ counterclockwise revolutions **18.** $2\frac{1}{3}$ counterclockwise revolutions

Sketch each angle in standard position when $n = 0$, $n = 1$, $n = 2$, and $n = -1$.

19. $45° + n \cdot 360°$ **20.** $-30° + n \cdot 360°$ **21.** $60° + n \cdot 180°$ **22.** $-30° + n \cdot 180°$

For Exercises 23–30:

a. Write a formula for the measures of all angles coterminal with the given angle.

b. Use the formula to find two angles, one positive and one negative, that are coterminal with the given angle.

23. 35° **24.** 140° **25.** −100° **26.** −210°

27. 520° **28.** 355° **29.** 1000° **30.** −3605°

Express in decimal degrees to the nearest tenth of a degree.

31. $15°30'$ **32.** $47°36'$ **33.** $72°50'$ **34.** $51°20'$

Express in decimal degrees to the nearest hundredth of a degree.

35. $25°45'$ **36.** $33°15'$ **37.** $45°18'20''$ **38.** $0°42'30''$

Express in degrees and minutes to the nearest minute.

39. $25.4°$ **40.** $63.6°$ **41.** $44.9°$ **42.** $27.1°$

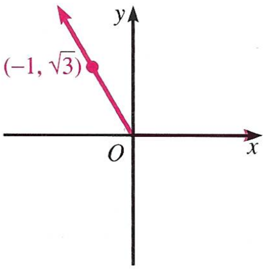
Express in degrees, minutes, and seconds to the nearest second.

43. $34.41°$ **44.** $18.27°$ **45.** $23.67°$ **46.** $58.83°$

The terminal side of an angle in standard position passes through the given point. Draw the angle and use a protractor to estimate its measure. ($\sqrt{3} \approx 1.73$)

Sample 2 $(-1, \sqrt{3})$

Solution



The measure is about 120°. *Answer*

47. $(4, -4)$ **48.** $(3, 4)$ **49.** $(4, 3)$ **50.** $(-4, 4)$
51. $(\sqrt{3}, 1)$ **52.** $(-1, \sqrt{3})$ **53.** $(-1, -\sqrt{3})$ **54.** $(\sqrt{3}, -1)$

Find a first-quadrant angle θ, $0° < \theta < 90°$, for which an angle four times as large as θ will be in the given quadrant.

Sample 3 Find a first-quadrant angle θ, $0° < \theta < 90°$, for which an angle four times as large will be in the second quadrant.

Solution You want 4θ to be in quadrant II: $90° < 4\theta < 180°$

$$\therefore 22.5° < \theta < 45°$$

Since any angle θ between 22.5° and 45° is in quadrant I, and 4θ is in quadrant II, you can choose, for example, 30°. *Answer*

B **55.** the first quadrant **56.** the third quadrant **57.** the fourth quadrant

Find a first-quadrant angle θ, $0° < \theta < 90°$, for which an angle six times as large as θ will be in the given quadrant.

58. the second quadrant

59. the third quadrant

60. the fourth quadrant

Find an angle θ for which an angle half as large as θ will be in the given quadrant.

61. the first quadrant

62. the second quadrant

63. the third quadrant

Find an angle θ for which an angle one fifth as large as θ will be in the given quadrant.

64. the first quadrant

65. the second quadrant

66. the third quadrant

Mixed Review Exercises

Find the third term in the expansion of each binomial.

1. $(x + 2y)^5$

2. $(a - 3b)^8$

3. $(m^2 + n^3)^{13}$

Simplify.

4. $\sqrt{-3} \cdot \sqrt{-12}$

5. $\log_2 14 - \log_2 7$

6. $(2^{3/4} - 2^{1/4})^2$

7. $\ln \sqrt[3]{e^2}$

8. $(3 - 2i)(4 + i)$

9. $\sqrt{20} + \sqrt{45} - \sqrt{125}$

10. $\dfrac{-6(3 - 5)}{(-7 + 9)^2}$

11. $\sqrt{\dfrac{98}{27}}$

12. $\log_3 3\sqrt{3}$

Calculator Key-In

Many scientific calculators have keys that convert between degrees, minutes, and seconds and decimal degrees. On some calculators there is a conversion key (usually labeled ° ′ ″) that allows you to convert, say, 32°45′10″ to decimal degrees as follows: enter 32, press the key; enter 45, press the key again; enter 10, press the key a third time. The result is 32.7527778°.

To convert from decimal degrees to degrees, minutes, and seconds, you just enter the decimal degrees and press the inverse conversion key. (If there is no such key, press the inverse key and then the conversion key.)

On other calculators you may be required to enter 32°45′10″ differently and the keys may be labeled with "DMS" for degrees-minutes-seconds. Test your calculator using 32°45′10″ and its decimal equivalent 32.7527778° before doing the following exercises.

Convert each measure to degrees, minutes, and seconds or to decimal degrees.

1. 58.6°

2. 29.7°

3. 86.43°

4. 108.26°

5. 36°25′36″

6. 45°11′19″

7. 73°52′25″

8. 115°42′51″

12-2 Trigonometric Functions of Acute Angles

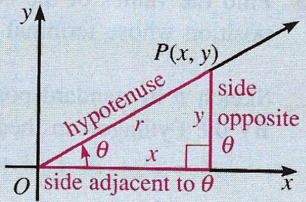
Objective To define trigonometric functions of acute angles.

To define the trigonometric functions of an acute angle θ, first place θ in standard position as shown in the diagram below. Next, choose any point $P(x, y)$, other than the origin, on the terminal side of θ and let r be the distance OP. Then the following definitions can be made:

The **sine** of θ, written $\sin\theta$, is equal to $\frac{y}{r}$.

The **cosine** of θ, written $\cos\theta$, is equal to $\frac{x}{r}$.

The **tangent** of θ, written $\tan\theta$, is equal to $\frac{y}{x}$.



The ratios $\frac{y}{r}$, $\frac{x}{r}$, and $\frac{y}{x}$ depend only on θ and not on the choice of P. (See Exercise 32.) Therefore, the sine, cosine, and tangent are functions of θ. Notice that x, y, and r are the lengths of the legs and hypotenuse, respectively, of right triangle OMP. The definitions above can be restated in terms of these lengths, for any right triangle containing the acute angle θ.

$$\sin\theta = \frac{y}{r} = \frac{\text{length of side opposite }\theta}{\text{length of the hypotenuse}}$$

$$\cos\theta = \frac{x}{r} = \frac{\text{length of side adjacent to }\theta}{\text{length of the hypotenuse}}$$

$$\tan\theta = \frac{y}{x} = \frac{\text{length of side opposite }\theta}{\text{length of side adjacent to }\theta}$$

Example 1 An acute angle θ is in standard position and its terminal side passes through $P(4, 5)$. Find $\sin\theta$, $\cos\theta$, and $\tan\theta$.

Solution Here $x = 4$ and $y = 5$. To find r, use the Pythagorean theorem (page 401):

$$r^2 = 4^2 + 5^2 = 16 + 25 = 41, \quad \text{so} \quad r = \sqrt{41}$$

$$\therefore \sin\theta = \frac{y}{r} = \frac{5}{\sqrt{41}} = \frac{5\sqrt{41}}{41}; \cos\theta = \frac{x}{r} = \frac{4}{\sqrt{41}} = \frac{4\sqrt{41}}{41}; \tan\theta = \frac{5}{4}$$

The three remaining trigonometric functions are **reciprocal functions** of those functions already defined.

The **cotangent** of θ, written $\cot\theta$, is equal to $\frac{x}{y}$, or $\frac{1}{\tan\theta}$.

The **secant** of θ, written $\sec\theta$, is equal to $\frac{r}{x}$, or $\frac{1}{\cos\theta}$.

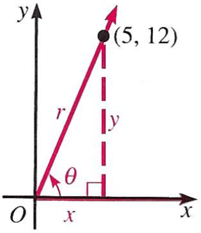
The **cosecant** of θ, written $\csc\theta$, is equal to $\frac{r}{y}$, or $\frac{1}{\sin\theta}$.

Example 2 Find the values of the six trigonometric functions of an angle θ in standard position whose terminal side passes through (5, 12).

Solution Sketch θ in standard position.
By the Pythagorean theorem:

$$r^2 = x^2 + y^2 = 5^2 + 12^2 = 169$$

$$r = 13$$



$$\sin\theta = \frac{y}{r} = \frac{12}{13} \qquad \csc\theta = \frac{r}{y} = \frac{13}{12}$$

$$\cos\theta = \frac{x}{r} = \frac{5}{13} \qquad \sec\theta = \frac{r}{x} = \frac{13}{5}$$

$$\tan\theta = \frac{y}{x} = \frac{12}{5} \qquad \cot\theta = \frac{x}{y} = \frac{5}{12}$$

An equation involving trigonometric functions of an angle θ that is true for all values of θ is a *trigonometric identity*. Using the diagram above and the definitions of this lesson, notice that

$$\frac{\sin\theta}{\cos\theta} = \frac{\frac{y}{r}}{\frac{x}{r}} = \frac{y}{x} = \tan\theta \quad \text{and} \quad \frac{\cos\theta}{\sin\theta} = \frac{\frac{x}{r}}{\frac{y}{r}} = \frac{x}{y} = \cot\theta.$$

Therefore, the following equations are identities:

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

The next identity is called a *Pythagorean identity*. Notice that we write $\sin^2\theta$ for $(\sin\theta)^2$ and $\cos^2\theta$ for $(\cos\theta)^2$.

$$\sin^2\theta + \cos^2\theta = 1$$

To prove this identity, you use the Pythagorean relationship $x^2 + y^2 = r^2$.

$$\sin^2\theta + \cos^2\theta = \left(\frac{y}{r}\right)^2 + \left(\frac{x}{r}\right)^2 = \frac{y^2}{r^2} + \frac{x^2}{r^2} = \frac{y^2 + x^2}{r^2} = \frac{r^2}{r^2} = 1$$

Example 3 Find $\cos\theta$ and $\tan\theta$ if θ is an acute angle and $\sin\theta = \frac{1}{3}$.

Solution 1 Use the identity $\sin^2\theta + \cos^2\theta = 1$.

$$\left(\frac{1}{3}\right)^2 + \cos^2\theta = 1$$

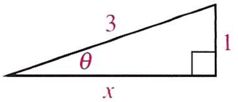
$$\cos^2\theta = 1 - \left(\frac{1}{3}\right)^2 = 1 - \frac{1}{9} = \frac{8}{9}$$

$$\cos\theta = \sqrt{\frac{8}{9}} = \frac{2\sqrt{2}}{3} \quad \textbf{\textit{Answer}}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\frac{1}{3}}{\frac{2\sqrt{2}}{3}} = \frac{1}{2\sqrt{2}} = \frac{\sqrt{2}}{4} \quad \textbf{\textit{Answer}}$$

Solution 2 Make a diagram showing

$$\sin\theta = \frac{\text{length of side opposite } \theta}{\text{length of the hypotenuse}} = \frac{1}{3}.$$

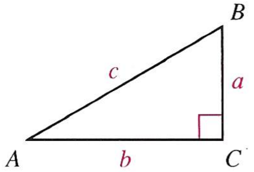


Then find x: $x^2 = 3^2 - 1^2 = 8$

$x = 2\sqrt{2}$

Then $\cos\theta = \frac{2\sqrt{2}}{3}$ and $\tan\theta = \frac{1}{2\sqrt{2}} = \frac{\sqrt{2}}{4}$. ***Answer***

The sine and cosine are called **cofunctions.** Other pairs of cofunctions are the tangent and cotangent and the secant and cosecant. There are six cofunction identities that can be derived from right triangle ABC.



$\sin A = \frac{a}{c} = \cos B$ $\qquad$ $\cos A = \frac{b}{c} = \sin B$

$\tan A = \frac{a}{b} = \cot B$ $\qquad$ $\cot A = \frac{b}{a} = \tan B$

$\sec A = \frac{c}{b} = \csc B$ $\qquad$ $\csc A = \frac{c}{a} = \sec B$

Notice that $\angle A$ and $\angle B$ are complementary; that is, $m\angle A + m\angle B = 90°$. Notice also that:

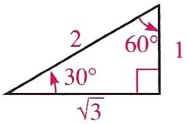
$\sin\theta = \cos(90° - \theta)$ $\qquad$ $\cos\theta = \sin(90° - \theta)$

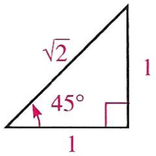
$\tan\theta = \cot(90° - \theta)$ $\qquad$ $\cot\theta = \tan(90° - \theta)$

$\sec\theta = \csc(90° - \theta)$ $\qquad$ $\csc\theta = \sec(90° - \theta)$

Thus, any trigonometric function of an acute angle is equal to the cofunction of the complement of the angle. Since 30° angles and 60° angles are complementary, they illustrate the cofunction relationship. For example, $\tan 60° = \cot 30°$.

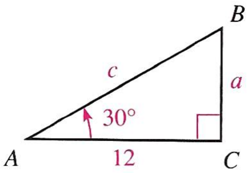
You can read the trigonometric functions of 30°, 45°, and 60° from the triangles shown below. To recall these values, you may find it easier to draw the triangles rather than to memorize the table.





θ	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\csc\theta$	$\sec\theta$	$\cot\theta$
30°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	2	$\frac{2\sqrt{3}}{3}$	$\sqrt{3}$
45°	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	$\sqrt{2}$	$\sqrt{2}$	1
60°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{2\sqrt{3}}{3}$	2	$\frac{\sqrt{3}}{3}$

Example 4 Use the diagram at the right. Find the lengths of side $\overline{BC}$ and side $\overline{AB}$.



Solution

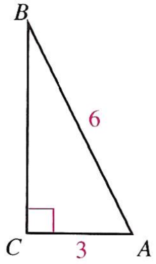
$$\tan 30° = \frac{BC}{AC} \qquad \cos 30° = \frac{AC}{AB}$$

$$\frac{\sqrt{3}}{3} = \frac{a}{12} \qquad \frac{\sqrt{3}}{2} = \frac{12}{c}$$

$$3a = 12\sqrt{3} \qquad \sqrt{3}c = 24$$

$$a = 4\sqrt{3} \qquad c = \frac{24}{\sqrt{3}} = 8\sqrt{3}$$

Example 5 Use the diagram at the right to find the measure of $\angle A$.



Solution

$$\cos A = \frac{AC}{AB} = \frac{3}{6} = \frac{1}{2}$$

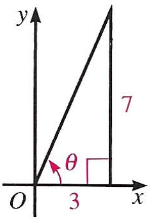
$$\cos 60° = \frac{1}{2}$$

$\therefore \angle A = 60°.$ ***Answer***

Oral Exercises

Give the values of the six trigonometric functions of θ.

1.



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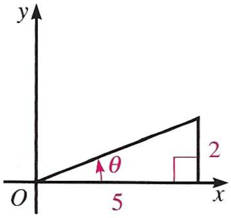

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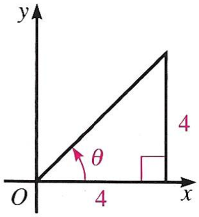

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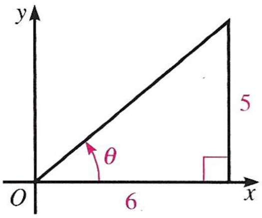


Use the cofunction identities to find the measure of the acute angle θ.

5. $\sin\theta = \cos 25°$

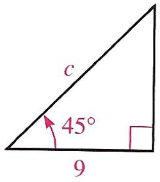
6. $\tan\theta = \cot 70°$

7. $\sec\theta = \csc 15°$

8. $\cos\theta = \sin 45°$

The length of one side of a right triangle is given. Give the lengths of the other two sides.

9.



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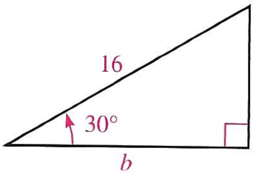

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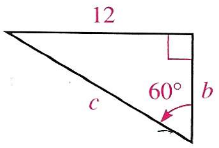
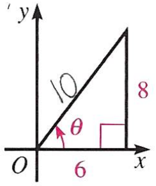


12. If $\sin\theta = \frac{2}{3}$ for an acute angle θ, state two ways to find $\cos\theta$ and $\tan\theta$.

Written Exercises

Find the values of the six trigonometric functions of angle θ.

A **1.**



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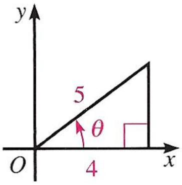

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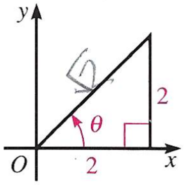

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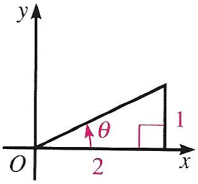


Find the values of the six trigonometric functions of an angle θ in standard position whose terminal side passes through point P.

5. $P(8, 15)$

6. $P(3, 4)$

7. $P(3, 1)$

8. $P(5, 5)$

Complete the table. In each case, θ is an acute angle.

	9.	10.	11.	12.	13.	14.	15.
$\sin\theta$	$\frac{3}{5}$	?	?	$\frac{\sqrt{2}}{3}$	$\frac{\sqrt{161}}{15}$	?	?
$\cos\theta$	?	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	?	?	$\frac{5}{13}$	$\frac{1}{4}$
$\tan\theta$	?	?	?	?	?	?	?

Use the cofunction identities to find the measure of the acute angle ϕ.

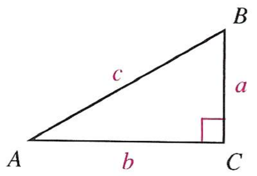
16. $\sin\phi = \cos 50°$

17. $\cos\phi = \sin 40°$

18. $\tan\phi = \cot 17°$

19. $\sec\phi = \csc 80°$

In Exercises 20–25, use the diagram at the right. Find the lengths of the sides and the measures of the angles that are not given. Leave your answers in simplest radical form.



20. $a = 6$, $\angle A = 30°$

21. $b = 2$, $\angle A = 45°$

22. $c = 10$, $\angle A = 45°$

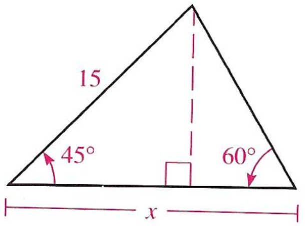
23. $c = 20$, $\angle A = 60°$

24. $a = 3$, $c = 6$

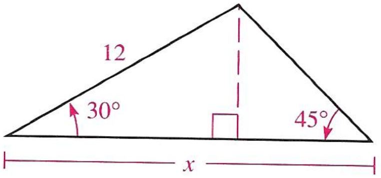
25. $a = 4$, $b = 4$

In Exercises 26–29, find the length of x. Leave your answers in simplest radical form.

B **26.**



27.



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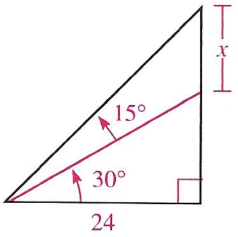

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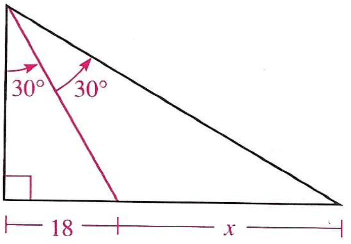


30. Prove that $\cot \theta = \dfrac{\cos \theta}{\sin \theta}$.

31. The three *Pythagorean identities* in trigonometry are as follows:

$\sin^2 \theta + \cos^2 \theta = 1 \qquad \tan^2 \theta + 1 = \sec^2 \theta \qquad 1 + \cot^2 \theta = \csc^2 \theta$

We have shown the first of these to be true for every acute angle θ. Use this first Pythagorean identity to prove that the other two identities are true for every acute angle θ.

32. $P(x, y)$ and $Q(x', y')$ are two points on the terminal side of an acute angle θ in standard position. Let $OP = r$ and $OQ = s$. Use similar triangles to show that $\dfrac{y}{r} = \dfrac{y'}{s}$, $\dfrac{x}{r} = \dfrac{x'}{s}$, and $\dfrac{y}{x} = \dfrac{y'}{x'}$. (This shows that using point Q to define the trigonometric functions θ will yield the same values as using point P.)

C **33.** Angle θ is acute and $\sin \theta = u$. Express the other five trigonometric functions of θ in terms of u. (*Hint:* Recall that $\sin^2 \theta + \cos^2 \theta = 1$.)

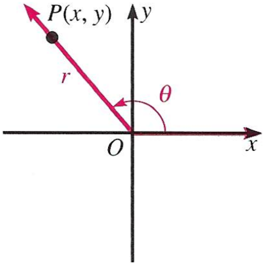
34. Angle ϕ is acute and $\cos \phi = v$. Express the other five trigonometric functions of ϕ in terms of v.

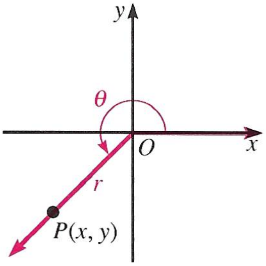
12-3 Trigonometric Functions of General Angles

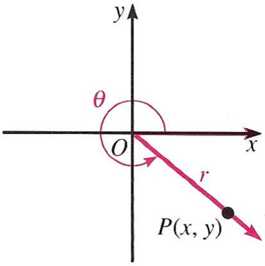
Objective To define trigonometric functions of general angles.

In Lesson 12-2, you learned the definitions for the six trigonometric functions for acute angles. Now you'll extend these definitions to angles of any measure.

As you did with acute angles, place the angle θ in standard position, as shown in each diagram below. Choose a point $P(x, y)$ on the terminal side of θ in each quadrant and let r be the distance OP.







For any angle θ and any point (x, y) on the terminal side:

$$\sin\theta = \frac{y}{r} \qquad \cos\theta = \frac{x}{r} \qquad \tan\theta = \frac{y}{x},\text{ if } x \neq 0$$

$$\csc\theta = \frac{r}{y},\text{ if } y \neq 0 \qquad \sec\theta = \frac{r}{x},\text{ if } x \neq 0 \qquad \cot\theta = \frac{x}{y},\text{ if } y \neq 0$$

Example 1 Find the values of the six trigonometric functions of an angle θ in standard position whose terminal side passes through $(8, -15)$.

Solution First make a sketch similar to the third one above.
Here $x = 8$ and $y = -15$. Since r is the distance between $(0, 0)$ and $(8, -15)$, you can find r by using the distance formula (page 402).

$$r = \sqrt{8^2 + (-15)^2} = \sqrt{64 + 225} = \sqrt{289} = 17$$

$$\sin\theta = \frac{y}{r} = -\frac{15}{17} \qquad \cos\theta = \frac{x}{r} = \frac{8}{17} \qquad \tan\theta = \frac{y}{x} = -\frac{15}{8}$$

$$\csc\theta = \frac{r}{y} = -\frac{17}{15} \qquad \sec\theta = \frac{r}{x} = \frac{17}{8} \qquad \cot\theta = \frac{x}{y} = -\frac{8}{15}$$

In the definitions above, r is always positive. Therefore, the signs of the functions of θ are determined by the signs of x and y, and these signs depend only on the quadrant of the terminal side of θ, as shown on the next page.

Function value	Quadrant of θ			
	I	II	III	IV
$\sin\theta$ $\csc\theta$	+	+	−	−
$\cos\theta$ $\sec\theta$	+	−	−	+
$\tan\theta$ $\cot\theta$	+	−	+	−

The sine and cosine functions are defined for all angles, but the other four functions are undefined for certain quadrantal angles.

Example 2 Determine which functions are defined for a 180° angle and find their values.

Solution Place the angle in standard position. Choose a point on the terminal side, such as $P(-1, 0)$. Then $x = -1$, $y = 0$, and $r = 1$.

$$\sin 180^\circ = \frac{y}{r} = \frac{0}{1} = 0$$

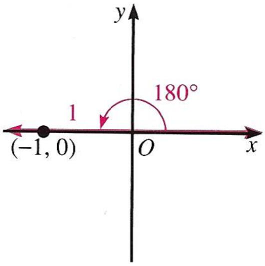
$$\cos 180^\circ = \frac{x}{r} = \frac{-1}{1} = -1$$

$$\tan 180^\circ = \frac{y}{x} = \frac{0}{-1} = 0$$

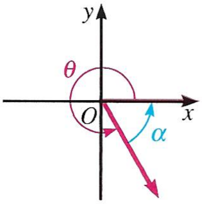
$\csc 180^\circ = \frac{r}{y}$ is undefined since $y = 0$.

$$\sec 180^\circ = \frac{r}{x} = \frac{1}{-1} = -1$$

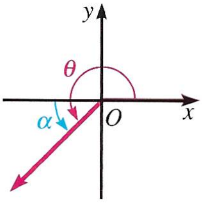
$\cot 180^\circ = \frac{x}{y}$ is undefined since $y = 0$.



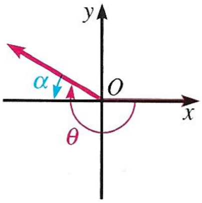
If θ is not a quadrantal angle, there is a unique acute angle α, corresponding to θ, formed by the terminal side of θ and the positive or negative x-axis. When θ is in standard position, we call α the **reference angle** of θ. Diagrams like those below are helpful in finding reference angles.



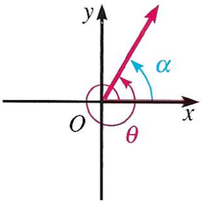
$\theta = 300^\circ$
$\alpha = 60^\circ$



$\theta = 225^\circ$
$\alpha = 45^\circ$



$\theta = -210^\circ$
$\alpha = 30^\circ$



$\theta = 420^\circ$
$\alpha = 60^\circ$

Example 3 Find the measure of the reference angle α for each given angle θ.

a. $\theta = 140°$ **b.** $\theta = 300°10'$ **c.** $\theta = -135°$

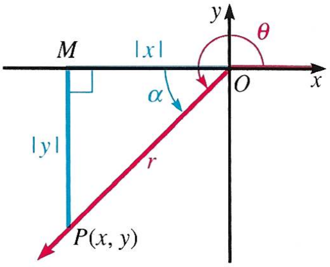
Solution

a. Since θ is in quadrant II: $\alpha = 180° - \theta = 180° - 140° = 40°$

b. Since θ is in quadrant IV:
$$\begin{aligned}\alpha &= 360° - \theta\\ &= 360° - 300°10'\\ &= 359°60' - 300°10' = 59°50'\end{aligned}$$

c. Find the positive angle that is coterminal with $-135°$:
$$-135° + 360° = 225°$$
Since 225° is in quadrant III: $\alpha = \theta - 180° = 225° - 180° = 45°$

You can use the reference angle to draw a **reference triangle** for any angle θ and then use the reference triangle to find values for the trigonometric functions of θ. For example, from right triangle OMP shown at the right, you see that $\cos \alpha = \frac{|x|}{r}$.



Since $\cos \theta = \frac{x}{r}$, you have
$$\cos \theta = \frac{x}{r} = \pm\frac{|x|}{r} = \pm\cos \alpha.$$

Similar results hold for the other trigonometric functions and for angles in other quadrants. Therefore, each trigonometric function of θ is either equal to the same function of α or equal to its opposite. That is, $\sin \theta = \pm\sin \alpha$, $\cos \theta = \pm\cos \alpha$, and so forth, where α is the reference angle of θ. The correct sign is determined by the quadrant of θ.

Example 4 Write cos 200° as a function of an acute angle.

Solution The reference angle of 200° is $200° - 180° = 20°$.
Since 200° is a third-quadrant angle, its cosine is negative.
$\therefore \cos 200° = -\cos 20°$. ***Answer***

Example 5 Find the exact value of the following.

a. tan 330° **b.** csc (−225°)

Solution

a. The reference angle of 330° is 30°.
Since 330° is a fourth-quadrant angle, its tangent is negative.
$\therefore \tan 330° = -\tan 30° = -\frac{\sqrt{3}}{3}$ ***Answer***

b. The reference angle of −225° is 45°.
Since −225° is a second-quadrant angle, its cosecant is positive.
$\therefore \csc(-225°) = \csc 45° = \sqrt{2}$ ***Answer***

The identities proved for acute angles in Lesson 12-2 are true for general angles θ as well. If $\cos\theta \neq 0$, you have

$$\frac{\sin\theta}{\cos\theta} = \frac{\frac{y}{r}}{\frac{x}{r}} = \frac{y}{x} = \tan\theta \quad \text{and} \quad \frac{\cos\theta}{\sin\theta} = \frac{\frac{x}{r}}{\frac{y}{r}} = \frac{x}{y} = \cot\theta.$$

And, if $P(x, y)$ is at a distance r from the origin on the terminal side of θ, the distance formula gives

$$r = \sqrt{x^2 + y^2} \quad \text{or} \quad r^2 = x^2 + y^2.$$

Therefore, for any angle θ,

$$\sin^2\theta + \cos^2\theta = \frac{y^2 + x^2}{r^2} = 1.$$

Example 6 Find the five other trigonometric functions of θ if $\cos\theta = -\frac{2}{5}$ and $180° < \theta < 360°$.

Solution 1 Use $\sin^2\theta + \cos^2\theta = 1$.

$$\sin^2\theta + \left(-\frac{2}{5}\right)^2 = 1$$

$$\sin^2\theta = 1 - \frac{4}{25} = \frac{21}{25}$$

$$\sin\theta = \pm\sqrt{\frac{21}{25}}$$

$$\sin\theta = \frac{\sqrt{21}}{5} \quad \text{or} \quad \sin\theta = -\frac{\sqrt{21}}{5}$$

Since $\cos\theta$ is negative and $180° < \theta < 360°$, θ is a third-quadrant angle and $\sin\theta$ is negative.

$$\sin\theta = -\frac{\sqrt{21}}{5}$$

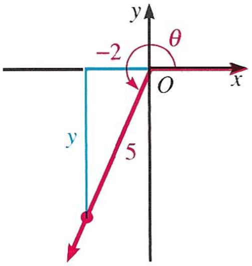
$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{-\frac{\sqrt{21}}{5}}{-\frac{2}{5}} = \frac{\sqrt{21}}{2}$$

The three remaining functions are reciprocals of those already known:

$$\sec\theta = -\frac{5}{2} \qquad \csc\theta = -\frac{5}{\sqrt{21}} = -\frac{5\sqrt{21}}{21} \qquad \cot\theta = \frac{2}{\sqrt{21}} = \frac{2\sqrt{21}}{21}$$

Solution 2 If $180° < \theta < 360°$ and $\cos\theta$ is negative, θ must be a third-quadrant angle.

Make a sketch as shown.



The Pythagorean theorem gives:
$y^2 = 5^2 - (-2)^2 = 21$ or $y = -\sqrt{21}$

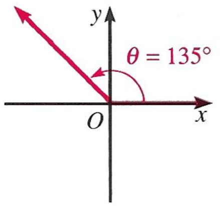
Thus, $\sin\theta = -\frac{\sqrt{21}}{5}$ and $\tan\theta = \frac{-\sqrt{21}}{-2} = \frac{\sqrt{21}}{2}$.

The remaining functions are obtained as in Solution 1.

Oral Exercises

State the measure of the reference angle α for each angle θ.

1.



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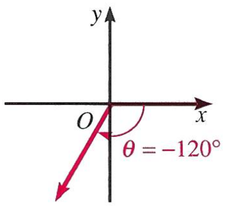

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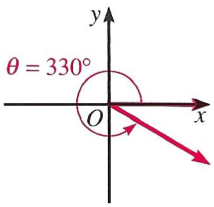

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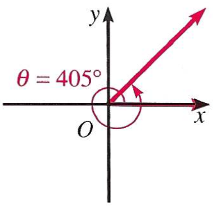


For each angle θ, name its quadrant and its reference angle.

5. $\theta = 170°$

6. $\theta = 250°$

7. $\theta = 210°$

8. $\theta = -70°$

9. $\theta = -305°$

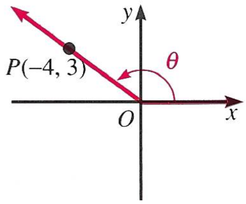
10. $\theta = -200°$

11. $\theta = 460°$

12. $\theta = -400°$

Give the values of the six trigonometric functions of θ. If any value is undefined, say so.

13.



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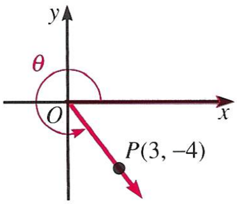

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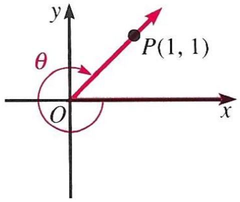

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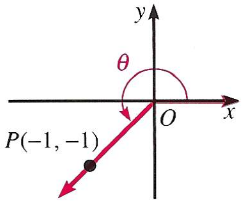

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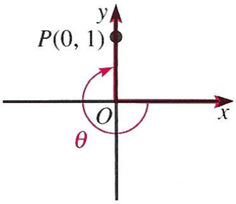

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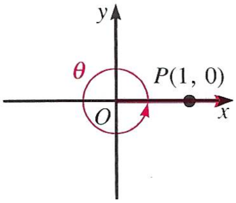

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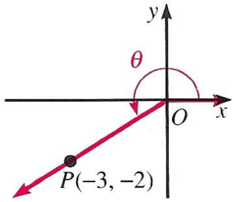

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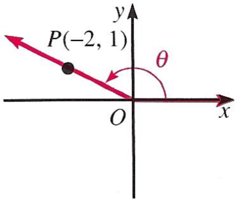


In Exercises 21–28, name the quadrant of θ. (Use the chart on page 562.)

21. $\sin \theta < 0, \cos \theta > 0$

22. $\sin \theta < 0, \cos \theta < 0$

23. $\sin \theta > 0, \cos \theta > 0$

24. $\sin \theta > 0, \cos \theta < 0$

25. $\sec \theta > 0, \tan \theta < 0$

26. $\csc \theta < 0, \cos \theta < 0$

27. $\cos \theta < 0, 180° < \theta < 360°$

28. $\tan \theta > 0, 90° < \theta < 270°$

29. If $\cos \theta = \frac{1}{4}$, $\sec \theta =$ __?__

30. If $\sin \theta = -\frac{3}{5}$, $\csc \theta =$ __?__

31. If $\cot \theta = -3$, $\tan \theta =$ __?__

32. If $\sec \theta = 1.5$, $\cos \theta =$ __?__

Written Exercises

Find the values of the six trigonometric functions of an angle θ in standard position whose terminal side passes through point P.

A **1.** cos $P(3, -4)$ sin **2.** $P(-5, 12)$ **3.** $P(-7, 24)$ **4.** $P(-8, -15)$

Complete the table. If any value is undefined, say so.

		$\sin\theta$	$\cos\theta$	$\tan\theta$	$\sec\theta$	$\csc\theta$	$\cot\theta$
5.	0°	?	?	?	?	?	?
6.	90°	?	?	?	?	?	?
7.	180°	?	?	?	?	?	?
8.	270°	?	?	?	?	?	?

Find the measure of the reference angle α of the given angle θ.

9. $\theta = 233°$ **10.** $\theta = 126°$ **11.** $\theta = -205°$ **12.** $\theta = -112°$

13. $\theta = 512°$ **14.** $\theta = 659°$ **15.** $\theta = -725°$ **16.** $\theta = -611°$

17. $\theta = 96.4°$ **18.** $\theta = 134.7°$ **19.** $\theta = -184.1°$ **20.** $\theta = -344.2°$

21. $\theta = 156°20'$ **22.** $\theta = 213°40'$ **23.** $\theta = 152°30'$ **24.** $\theta = 312°50'$

Write each of the following as a function of an acute angle.

25. $\cos 216°$ **26.** $\tan 334°$ **27.** $\sin(-17°)$

28. $\sec(-106°)$ **29.** $\cot 287.1°$ **30.** $\csc 117.8°$

31. $\cos(-221.9°)$ **32.** $\sin(-46.6°)$ **33.** $\tan 265°20'$

34. $\cot 137°10'$ **35.** $\sin 212°40'$ **36.** $\sin 324°30'$

Find the exact value of the six trigonometric functions of each angle.

37. 330° **38.** 240° **39.** 135° **40.** −60°

41. 315° **42.** 300° **43.** −150° **44.** 480°

First give the quadrant of angle θ. Then find the five other trigonometric functions of θ. Give answers involving radicals in simplest radical form.

B **45.** $\cos\theta = -\frac{8}{17}$, $0° < \theta < 180°$ **46.** $\sin\theta = -\frac{4}{5}$, $90° < \theta < 270°$

47. $\sin\theta = -\frac{5}{13}$, $\cos\theta > 0$ **48.** $\cos\theta = -\frac{3}{5}$, $\sin\theta > 0$

49. $\cos\theta = \frac{2}{3}$, $0° < \theta < 270°$ **50.** $\sin\theta = -\frac{2}{5}$, $-90° < \theta < 180°$

51. $\tan\theta = -\frac{3}{4}$, $\cos\theta < 0$ **52.** $\sec\theta = \frac{13}{5}$, $\sin\theta < 0$

Name all angles θ, $0° \leq \theta < 360°$, that make the statement true.

53. $\sin \theta = 0$
54. $\cos \theta = 0$
55. $\tan \theta = 0$
56. $\sin \theta = 1$
57. $\tan \theta = 1$
58. $\cos \theta = -1$
59. $\sin \theta = -1$
60. $\sin \theta = \frac{1}{2}$
61. $\cos \theta = -\frac{\sqrt{3}}{2}$
62. $\sin \theta = \sin 300°$
63. $\tan \theta = \tan 15°$
64. $\sin \theta = \sin 40°$
65. $\cos \theta = \cos 70°$
66. $\cos \theta = \cos 100°$
67. $\tan \theta = \tan 20°$

Write each of the following in terms only of $\sin \theta$ and $\cos \theta$. (*Hint:* A sketch will be useful.)

C **68.** $\sin(-\theta)$
69. $\cos(-\theta)$
70. $\tan(-\theta)$
71. $\sin(180° + \theta)$
72. $\sin(180° - \theta)$
73. $\cos(180° + \theta)$
74. $\cos(180° - \theta)$
75. $\tan(180° + \theta)$
76. $\tan(180° - \theta)$

Mixed Review Exercises

Evaluate.

1. $\sum_{k=1}^{8} 5 \cdot 2^{k-1}$
2. $\sum_{k=1}^{20} (3k + 1)$
3. $\sum_{k=1}^{\infty} 10\left(\frac{3}{5}\right)^{k-1}$

Find the zeros of each function. If the function has no zeros, say so.

4. $f(x) = 3x + 5$
5. $g(x) = \log(x - 1)$
6. $h(x) = 2x^2 + x - 3$
7. $F(x) = \frac{x^3 - 4x}{x^2 + 1}$
8. $G(x) = x^2 - 4x + 5$
9. $H(x) = \left(\frac{1}{2}\right)^x$

Career Note / *Flight Engineer*

Modern aircraft are operated with the aid of many instruments. A flight engineer assists the pilot in checking the plane before takeoff, checks many of the instruments in flight, and records performance levels and fuel usage. Flight engineers report any mechanical problems to the pilot and sometimes make repairs. Flight engineers are licensed by the Federal Aviation Administration.

Most airlines look for candidates with a high school education and at least two years of college. Courses in mathematics and science are important.

12-4 Values of Trigonometric Functions

Objective To use a calculator or trigonometric tables to find values of trigonometric functions.

It is not possible to give *exact* trigonometric function values for most angles. You can, however, get good approximations using a scientific calculator. For example, to find cos 52°, enter 52 and press the cos key to obtain 0.6156615.

If a calculator is not available, you can use Table 4 or Table 5 on pages 814–826 to approximate the functions of an acute angle θ as follows:

1. If $0° \leq \theta \leq 45°$, find the function name in the *top* row and look *down* to the entry opposite the degree measure of θ at the extreme *left*.
2. If $45° \leq \theta \leq 90°$, find the function name in the *bottom* row and look *up* to the entry opposite the degree measure of θ at the extreme *right*.

Example 1 Find each function value to four significant digits.

a. sec 34.7°

b. cos 84°20′

Solution 1 Using a Calculator

a. Many scientific calculators have keys labeled *sin, cos,* and *tan,* but not *sec, csc,* and *cot*. So to find values for secant, cosecant, and cotangent, you may need to use reciprocal functions (page 556).

$$\sec 34.7° = \frac{1}{\cos 34.7°} = \frac{1}{0.822144} = 1.2163319$$

$\therefore$ to four significant digits, sec 34.7° = 1.216. ***Answer***

b. Some calculators operate only with decimal degrees. First you may need to change 84°20′ to decimal degrees by dividing 20′ by 60. (60′ = 1°)

$$\cos 84°20' = \cos 84.333333° = 0.0987408$$

$\therefore$ to four significant digits, cos 84°20′ = 0.0987. ***Answer***

Solution 2 Using Tables

a. In Table 4 look *down* the column under "sec θ" until you see the entry opposite 34.7° at the left. You should find "1.216."

$\therefore$ sec 34.7° = 1.216 ***Answer***

b. In Table 5 look *up* the column over "Cos" until you see the entry opposite 84°20′ at the right. You should find "0.0987."

$\therefore$ cos 84°20′ = 0.0987 ***Answer***

Even though the values given by Table 4 and Table 5 are approximate, it's a common practice to use =, as in Example 1, rather than ≈.

Using a calculator you can easily find function values for angles with measures like 64.72° and 26°27′. However, these measures fall between consecutive entries in the tables. To use the tables, you may either use the table entry nearest the given measure or use *linear interpolation* (Lesson 8-9).

Example 2 Find $\cos 26^\circ 27'$

Solution 1 Using a Calculator

You may need to change 26°27′ to decimal degrees by dividing 27′ by 60.

$\therefore \cos 26^\circ 27' = \cos 26.45^\circ = 0.8953234.$ ***Answer***

Solution 2 Using Tables

Notice that $\cos 26^\circ 20' > \cos 26^\circ 27' > \cos 26^\circ 30'$. Using a vertical arrangement for the linear interpolation, you can write:

$$\begin{array}{r|c|c|l}
 & \theta & \cos\theta & \\ \hline
10' \left[\; 7' \left[\right. \right. & 26^\circ 20' & 0.8962 & \left. \right] d \; \left. \right] -0.0013 \\
 & 26^\circ 27' & \underline{\;?\;} & \\
 & 26^\circ 30' & 0.8949 &
\end{array}
\left\{\begin{array}{l}\text{A negative number}\\ \text{because the function}\\ \text{value decreases.}\end{array}\right.$$

$$\frac{7}{10} = \frac{d}{-0.0013};\ d = \frac{7}{10}(-0.0013) = -0.0009$$

$\therefore \cos 26^\circ 27' = 0.8962 + (-0.0009) = 0.8953$ ***Answer***

Example 3 Find the measure of the acute angle θ to the nearest tenth of a degree when $\sin \theta = 0.8700$.

Solution 1 Using a Calculator

To find an acute angle when given one of its function values, you use the inverse function keys ($\sin^{-1}$, $\cos^{-1}$, $\tan^{-1}$, or inv sin, inv cos, inv tan).

If $\sin \theta = 0.8700$, then

$$\theta = \sin^{-1} 0.8700 = 60.458639.$$

$\therefore$ to the nearest tenth of a degree, $\theta = 60.5^\circ$. ***Answer***

Caution: $\sin^{-1} x$ is *a number whose sine is x*. The notation $\sin^{-1} x$ is not exponential notation.

It does *not* mean $\frac{1}{\sin x}$.

Solution 2 Using Tables

Reverse the process described in Example 1. Look in the sine columns of Table 4 until you find the entry nearest 0.8700. This entry is 0.8704, and it is opposite 60.5° on the right.

$\therefore$ to the nearest tenth of a degree, $\theta = 60.5^\circ$. ***Answer***

Sometimes you may be asked to find solutions to the nearest minute instead of to the nearest tenth of a degree. You can use a calculator or tables to do this. To use the tables, you again use linear interpolation.

Example 4 Find the measure of the acute angle θ to the nearest minute when $\cot \theta = 0.0782$.

Solution 1 Using a Calculator

Since $\cot \theta = 0.0782$, $\tan \theta = \frac{1}{0.0782} = 12.787724$. Therefore,

$$\theta = \tan^{-1} 12.787724 = 85.52857^\circ.$$

To convert 0.52857° to minutes, multiply by 60 to obtain $31.714191'$.
$\therefore$ to the nearest minute, $\theta = 85^\circ 32'$. ***Answer***

Solution 2 Using Tables

First locate in Table 5 the nearest values for $\cot \theta$ that are above and below 0.0782. Then arrange the values as follows.

$$\begin{array}{c|l} \theta & \cot\theta \\ \hline \end{array}$$

$$10'\left[\begin{array}{c} 85^\circ 40' \\ d'\left[\begin{array}{c} ? \\ 85^\circ 30' \end{array}\right. \end{array}\right. \left|\begin{array}{l} 0.0758 \\ \left.\begin{array}{l} 0.0782 \\ 0.0787 \end{array}\right] -0.0005 \end{array}\right] -0.0029 \left\{\begin{array}{l} \text{A negative number} \\ \text{because} \\ \cot 85^\circ 40' < \cot 85^\circ 30'. \end{array}\right.$$

$$\frac{d}{10} = \frac{-0.0005}{-0.0029};\ d = \frac{5}{29}(10) = 2$$

$\therefore$ to the nearest minute, $\theta = 85^\circ 30' + 2'$, or $85^\circ 32'$. ***Answer***

The trigonometric functions of *any* angle can be found with the help of a calculator or with reference angles.

Example 5 Find $\sin 147.8^\circ$ and $\cos 147.8^\circ$.

Solution 1 Using a Calculator

You can obtain the values directly.

$$\sin 147.8^\circ = 0.5329$$
$$\cos 147.8^\circ = -0.8462$$

Solution 2 Using Tables

The reference angle for 147.8° is $180^\circ - 147.8^\circ = 32.2^\circ$. Since 147.8° is a second-quadrant angle, its sine is positive and its cosine is negative. Therefore:

$$\sin 147.8^\circ = \sin 32.2^\circ = 0.5329$$
$$\cos 147.8^\circ = -\cos 32.2^\circ = -0.8462$$

Example 6 Find to the nearest tenth of a degree the measures of two angles satisfying $\cos\theta = -0.7455$ and $0° < \theta < 360°$.

Solution 1 Using a Calculator

You can follow the method of Solution 2 of Example 5 and first find the reference angle α of θ. Or you can work with θ directly:

$$\theta = \cos^{-1}(-0.7455) = 138.20206.$$

Therefore, one angle is 138.2°. (This is a second-quadrant angle.)
Since $\cos\theta < 0$, θ is a second- or third-quadrant angle. To find the other angle, first subtract 138.2° from 180° to obtain a reference angle of 41.8°. Then the third-quadrant angle is $\theta = 180° + 41.8° = 221.8°$.

$\therefore$ the angles are 138.2° and 221.8°. ***Answer***

Solution 2 Using Tables

First find the reference angle α of θ.
Since $\cos\alpha = 0.7455$, $\alpha = 41.8°$.
Because $\cos\theta < 0$, θ is a second- or third-quadrant angle.
Therefore,
$\theta_1 = 180° - \alpha = 180° - 41.8° = 138.2°$ and
$\theta_2 = 180° + \alpha = 180° + 41.8° = 221.8°$. ***Answer***

Oral Exercises

Use a calculator, Table 4, or Table 5 to find each function value to four significant digits.

1. $\sin 15°$

2. $\cos 42°$

3. $\tan 22.6°$

4. $\cot 34.7°$

5. $\cos 85.9°$

6. $\sin 58.2°$

7. $\csc 27°40'$

8. $\sec 49°20'$

9. $\cot 73°50'$

10. $\tan 56°10'$

11. $\sec 12°30'$

12. $\csc 47°20'$

Use a calculator or Table 4 to find the measure of acute angle *θ* to the nearest tenth of a degree.

13. $\cos\theta = 0.8960$

14. $\sin\theta = 0.6160$

15. $\cot\theta = 7.596$

16. $\tan\theta = 1.632$

17. $\sec\theta = 2.089$

18. $\csc\theta = 1.095$

Use a calculator or Table 5 to find the measure of acute angle *θ* to the nearest ten minutes.

19. $\sin\theta = 0.5324$

20. $\cos\theta = 0.8511$

21. $\tan\theta = 1.455$

22. $\cot\theta = 0.4950$

23. $\csc\theta = 1.046$

24. $\sec\theta = 5.164$

25. Explain how you would find the measures of two angles θ between 0° and 360° such that $\sin\theta = -0.5000$.

Written Exercises

Find each function value to four significant digits.

A **1.** $\tan 15.2^\circ$
2. $\cos 38.7^\circ$
3. $\sin 65.2^\circ$
4. $\cot 81.1^\circ$
5. $\cos 31^\circ 10'$
6. $\sin 46^\circ 30'$
7. $\sec 54^\circ 40'$
8. $\csc 23^\circ 20'$
9. $\csc 7.44^\circ$
10. $\sec 13.22^\circ$
11. $\cos 32.43^\circ$
12. $\sin 26.57^\circ$
13. $\cos 52^\circ 43'$
14. $\tan 29^\circ 36'$
15. $\cot 78^\circ 15'$
16. $\sin 87^\circ 53'$
17. $\sec 111.3^\circ$
18. $\csc 163.4^\circ$
19. $\sin 268.5^\circ$
20. $\cos 328.1^\circ$
21. $\cot 143^\circ 30'$
22. $\tan 173^\circ 20'$
23. $\cos 231^\circ 30'$
24. $\sin 312^\circ 40'$

Find the measure of the acute angle *θ* to the nearest tenth of a degree.

25. $\sin \theta = 0.3400$
26. $\cos \theta = 0.8400$
27. $\cot \theta = 1.700$
28. $\tan \theta = 1.325$
29. $\sec \theta = 3.555$
30. $\csc \theta = 3.000$

Find the measure of the acute angle *θ* to the nearest minute.

31. $\cos \theta = 0.8621$
32. $\sin \theta = 0.2654$
33. $\tan \theta = 0.1482$
34. $\cos \theta = 0.2715$
35. $\sin \theta = 0.7321$
36. $\tan \theta = 2.550$

Find the measures of two angles between 0° and 360° with the given function value. Give answers to the nearest tenth of a degree.

B **37.** $\sin \theta = 0.4875$
38. $\cos \theta = 0.7851$
39. $\tan \theta = -1.752$
40. $\sin \theta = -0.8300$
41. $\cos \theta = 0.2524$
42. $\tan \theta = 0.6182$

Find the measure of an angle *θ* between 0° and 360° that satisfies the stated conditions. Give answers to the nearest tenth of a degree.

43. $\cos \theta = 0.4275,\ 180^\circ < \theta < 360^\circ$
44. $\sin \theta = -0.5212,\ 90^\circ < \theta < 270^\circ$
45. $\sin \theta = -0.6118,\ \cos \theta > 0$
46. $\cos \theta = 0.7815,\ \tan \theta < 0$

Self-Test 1

Vocabulary

degree (p. 549)
initial side (p. 549)
terminal side (p. 549)
positive angle (p. 549)
negative angle (p. 549)
directed angle (p. 549)
standard position (p. 549)
quadrantal angle (p. 549)
coterminal angles (p. 550)
minute, second (p. 550)
sine (p. 555)
cosine (p. 555)
tangent (p. 555)
reciprocal functions (p. 556)
cotangent (p. 556)
secant (p. 556)
cosecant (p. 556)
identity (p. 556)
cofunction (p. 557)
reference angle (p. 562)

1. Name two angles, one positive and one negative, that are coterminal with the given angle. **Obj. 12-1, p. 549**

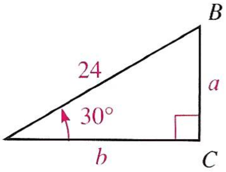
a. 250° **b.** −300°

2. Express 32°35′ in decimal degrees to the nearest hundredth of a degree.

3. Express 74.26′ in degrees, minutes, and seconds to the nearest second.

4. When angle θ is in standard position, its terminal side passes through $(\sqrt{7}, 3)$. Find the values of the six trigonometric functions of θ. **Obj. 12-2, p. 555**

5. In the right triangle shown at the right, find a, b, and the measure of $\angle B$.



6. Find the reference angle α for an angle with a measure of 300°. Find the exact values of the six trigonometric functions of 300°, using radicals when necessary. **Obj. 12-3, p. 561**

7. Find the following to four significant digits: **Obj. 12-4, p. 568**

a. cos 36°42′ **b.** csc 64.33°

8. Find the measure of angle θ to the nearest minute if θ is an acute angle and $\sin \theta = 0.8760$.

Check your answers with those at the back of the book.

Historical Note / *Tables of Sines*

Among the first historical appearances of trigonometry were ‘‘tables of chords,’’ recognizable as sine tables. Values for these early tables, such as the one devised by the Greek astronomer Hipparchus around 140 B.C., were found by using geometric methods to measure lengths of chords of a circle.

Modern trigonometric tables (and also the values found by a calculator or a computer) are computed by using terms of an infinite series. One such series is the *Taylor series* for sin x, where x is given in radian measure (see Lesson 13-1):

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ . \ . \ .$$

By adding up the terms of this series you can approximate the sine of a given angle to any desired accuracy. For example, using the four terms shown, you find that

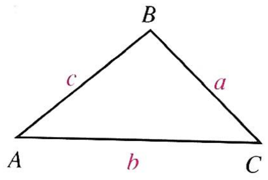
$$\sin 1 \approx 1 - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} \approx 0.84147,$$

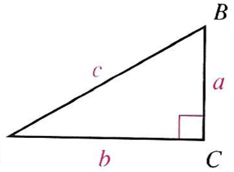
a value that is correct to five decimal places.

Triangle Trigonometry

12-5 Solving Right Triangles

Objective To find the sides and angles of a right triangle.





Any triangle *ABC* has six measurements associated with it: the lengths of the three sides, denoted by *a*, *b*, and *c*, and the measures of the three angles, denoted by $\angle A$, $\angle B$, and $\angle C$. (Note that the lower-case letters denote the lengths of sides opposite angles labeled by the corresponding capital letters.) If any three of these measurements (other than the three angle measures) are given, then the other three can be found.

Finding measurements for all the sides and angles of a triangle is called **solving the triangle.**

In the case of a right triangle *ABC*, one angle is 90°. You can solve the triangle if the lengths of two sides, or the length of one side and the measurement of one acute angle, are known.

Example 1 Solve the right triangle shown above if $\angle A = 36°$ and $b = 50$.

Solution 1 Since $\angle A + \angle B = 90°$, $\angle B = 90° - \angle A = 90° - 36° = 54°$.

$$\cot A = \frac{b}{a} \qquad \cos A = \frac{b}{c}$$

$$\cot 36° = \frac{50}{a} \qquad \cos 36° = \frac{50}{c}$$

$$a = \frac{50}{\cot 36°} \qquad c = \frac{50}{\cos 36°}$$

$$= \frac{50}{1.376} \qquad = \frac{50}{0.8090}$$

$$= 36.3 \qquad = 61.8$$

$\therefore \angle B = 54°$, $a = 36.3$, and $c = 61.8$ ***Answer***

Solution 2 Again $\angle B = 90° - 36° = 54°$.

$$\tan A = \frac{a}{b} \qquad \sec A = \frac{c}{b}$$

$$\tan 36° = \frac{a}{50} \qquad \sec 36° = \frac{c}{50}$$

$$a = 50(\tan 36°) \qquad b = 50(\sec 36°)$$

$$= 50(0.7265) \qquad = 50(1.236)$$

$$= 36.3 \qquad = 61.8$$

$\therefore \angle B = 54°$, $a = 36.3$, and $c = 61.8$ ***Answer***

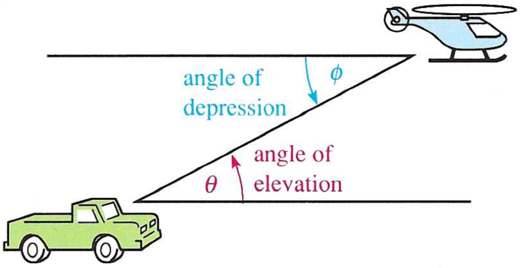
In Example 1, Solution 1 or Solution 2 are both well suited for calculator use. However, if tables are used, Solution 2 would be a better choice because long division is avoided.

Measurements of lengths and angles are approximations, as are the values given by tables and calculators. The following is a guide to the corresponding accuracies of length and angle measurements.

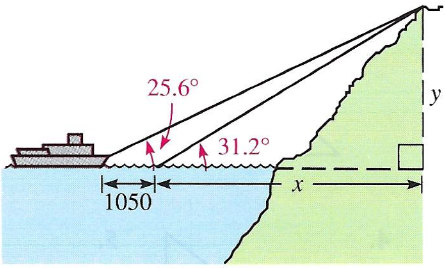
An angle measured to   A length measured to

$$\left.\begin{array}{rcr} & 1^\circ & \\ 0.1^\circ & \text{or} & 10' \\ 0.01^\circ & \text{or} & 1' \end{array}\right\} \text{ corresponds to } \left\{\begin{array}{l} 2 \text{ significant digits.} \\ 3 \text{ significant digits.} \\ 4 \text{ significant digits.} \end{array}\right.$$

The figure at the right shows a helicopter and a pickup truck. To see the helicopter, the driver's line of sight must be raised, or *elevated,* at an angle θ above the horizontal. This angle θ is called the **angle of elevation** of the helicopter. Similarly, the angle ϕ is called the **angle of depression** of the truck from the helicopter. The angle of elevation and the corresponding angle of depression always have the same measure.



Example 2 A research ship finds that the angle of elevation of a volcanic island peak is 25.6°. After the ship has moved 1050 m closer to the island, the angle of elevation is 31.2°. What is the height of the peak above sea level? (Note that the sketch shown at the right is not drawn to scale.)



Solution

$$\cot 31.2^\circ = \frac{x}{y} \qquad \text{and} \qquad \cot 25.6^\circ = \frac{1050 + x}{y}$$

$$x = y \cot 31.2^\circ \qquad\qquad y \cot 25.6^\circ = 1050 + x$$

$$x = 1.651y \qquad\qquad 2.087y = 1050 + x$$

Substitute $1.651y$ for x in the second equation.

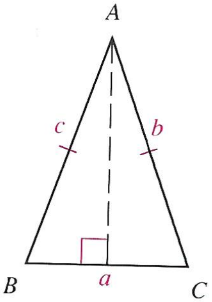
$$\begin{aligned} 2.087y &= 1050 + 1.651y \\ 0.436y &= 1050 \\ y &= \frac{1050}{0.436} = 2410 \text{ (to three significant digits)} \end{aligned}$$

$\therefore$ the volcanic peak is 2410 m above sea level. ***Answer***

The altitude drawn in the diagram at the right below divides isosceles triangle ABC into two congruent right triangles. Constructing such an altitude will help you to solve any isosceles triangle.

Example 3 Solve isosceles triangle ABC if $a = 31.0$ and $c = 42.5$.

Solution Make a sketch as shown.



In the right triangle on the left in the diagram, the side adjacent to $\angle B$ has length $\frac{1}{2}a$, or 15.5. So

$$\cos B = \frac{15.5}{42.5} = 0.3647 \text{ and } \angle B = 68.6°.$$

Since ΔABC is isosceles,

$$\angle C = \angle B = 68.6° \text{ and } b = c = 42.5$$

$$\angle A = 180° - (\angle B + \angle C)$$

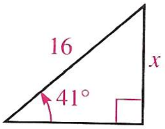
$$= 180° - 137.2° = 42.8°$$

$\therefore \angle B = 68.6°, \angle C = 68.6°, \angle A = 42.8°$, and $b = 42.5$ ***Answer***

Oral Exercises

Give an equation that can be used to find the value of x.

1.



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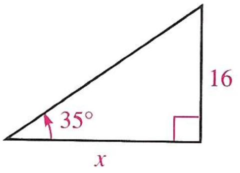

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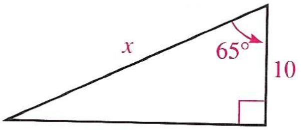

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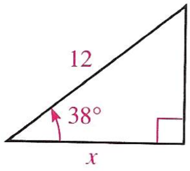

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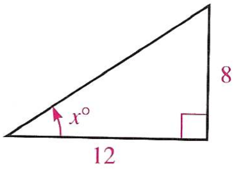

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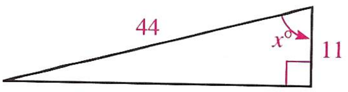


7. The angle of depression of Q from P measures 25°. Therefore, the angle of elevation of P from Q measures __?__.

8. What fact from geometry did you use in answering the question in Exercise 7?

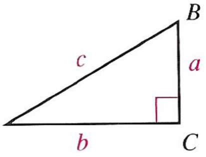
9. Give each angle to the nearest 10′.
a. 42°13′21″ **b.** 140°9′ **c.** 0°11′59″

10. Give each number to three significant digits.
a. 6.758 **b.** 1284.56 **c.** 0.39975

Written Exercises

Give lengths to three significant digits and angle measures to the nearest tenth of a degree or nearest ten minutes. You may wish to use a calculator.

Solve each right triangle *ABC*.



A **1.** $\angle A = 36.2°$, $c = 68$

2. $\angle B = 15.8°$, $c = 12.2$

3. $\angle B = 65.4°$, $a = 2.35$

4. $\angle A = 82.1°$, $b = 246$

5. $\angle B = 48.3°$, $b = 74.7$

6. $\angle A = 24.0°$, $a = 5.25$

7. $a = 230$, $c = 320$

8. $a = 52.5$, $b = 28.0$

9. $a = 0.123$, $b = 0.315$

10. $b = 3.90$, $c = 42.5$

11. $\angle B = 58°10'$, $c = 420$

12. $\angle A = 38°40'$, $c = 42.5$

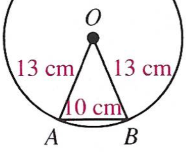
13. $\angle A = 15°30'$, $a = 4.50$

14. $\angle B = 67°20'$, $a = 450$

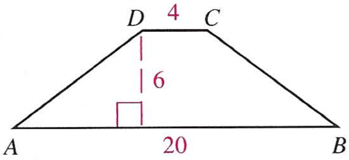
15. $\angle A = 30°50'$, $b = 53.5$

16. $\angle B = 85°10'$, $b = 0.620$

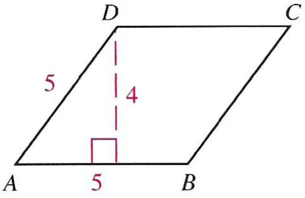
B **17.** The radius of circle O is 13 cm and the length of $\overline{AB}$ is 10 cm. Find the measure of $\angle AOB$.



Ex. 17



Ex. 18



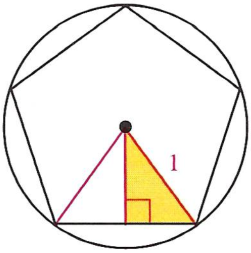
Ex. 19

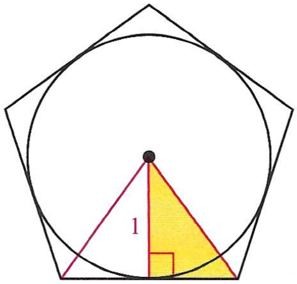
18. The height of an isosceles trapezoid is 6 units and the bases have lengths 4 units and 20 units. Find the measures of the angles.

19. A rhombus has sides 5 units long and its height is 4 units. Find its angles.

20. a. Find the perimeter of a regular pentagon inscribed in a unit circle.
b. Find the perimeter of a regular pentagon circumscribed about a unit circle.

(*Hint:* Use the shaded right triangles in the figures below.)





21. Repeat Exercise 20 for a regular polygon having 10 sides.

22. Repeat Exercise 20 for a regular polygon having 20 sides.

23. Explain why the number 2π lies between the answers to the (a) and (b) parts of Exercises 20, 21, and 22.

C **24.** **a.** Repeat Exercise 20 for a polygon having n sides.

b. Use the results of part (a) to tell what number is approached by $n \sin \left(\frac{180}{n}\right)^\circ$ as n gets larger and larger. What number is approached by $n \tan \left(\frac{180}{n}\right)^\circ$ as n gets larger and larger?

Problems

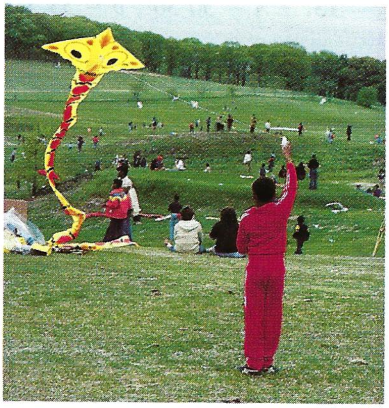
Give lengths to three significant digits and angle measures to the nearest tenth of a degree. You may wish to use a calculator.

A **1.** What is the angle of elevation of the sun when a tree 6.25 m tall casts a shadow 10.1 m long?

2. A boy flying a kite is standing 30 ft from a point directly under the kite. If the string to the kite is 50 ft long, find the angle of elevation of the kite.

3. A cable 4 m long is attached to a pole. The cable is staked to the ground 1.75 m from the base of the pole. Find the angle that the cable makes with the ground.

4. How far from the base of a building is the bottom of a 30 ft ladder that makes an angle of 75° with the ground?



5. The angle of elevation of the summit of a mountain from the bottom of a ski lift is 33°. A skier rides 1000 ft on this ski lift to get to the summit. Find the vertical distance between the bottom of the ski lift and the summit.

6. The approach pattern to an airport requires pilots to set an 11° angle of descent toward the runway. If a plane is flying at an altitude of 9500 m, at what distance (measured along the ground) from the airport must the pilot start the descent?

7. The distance from the point directly under a hot air balloon to the point where the balloon is staked to the ground with a rope is 285 ft. The angle of elevation up the rope to the balloon is 48°. Find the height of the balloon.

8. Opposite corners of a small rectangular park are joined by diagonal paths, each 360 m long. What are the dimensions of the park if the paths intersect at a 65° angle?

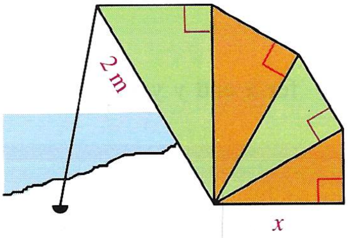
B **9.** A pendulum in a grandfather clock is 160 cm long. The horizontal distance between the farthest points in a complete swing is 65 cm. Through what angle does the pendulum swing?

10. A camping tent is supported by a rope stretched between two trees at a height of 210 cm. If the sides of the tent make an angle of 55° with the level ground, how wide is the tent at the bottom?

11. From the top of a 135 ft observation tower, a park ranger sights two forest fires on opposite sides of the tower. If their angles of depression are 42.5° and 32.6°, how far apart are the fires?

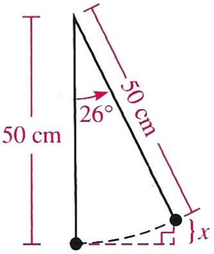
12. From a point 250 m from the base of a vertical cliff, the angles of elevation to the top and bottom of a radio tower on top of the cliff are 62.2° and 59.5°. How tall is the tower?

13. The side of the beach shelter shown at the right is made up of four 30°–60°–90° triangles. Find the dimension marked x.



Ex. 13

14. A pendulum 50 cm long is moved 26° from the vertical. How much is the lower end of the pendulum raised?

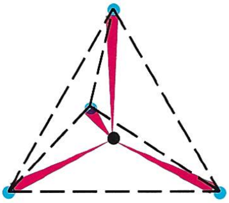


Ex. 14

C **15.** Two observers 1600 m apart on a straight, flat road measure the angles of elevation of a helicopter hovering over the road between them. If these angles are 32.0° and 50.5°, how high is the helicopter?

16. The pilot of a hot air balloon sees a field straight ahead that he knows is 1000 ft long. The angles of depression to the ends of the field are 25.4° and 34.7°. What is the height of the balloon and its horizontal distance from the nearer end of the field?

17. In a molecule of carbon tetrachloride, the four chlorine atoms are at the vertices of a regular tetrahedron, with the carbon atom in the center. What is the angle between two of the carbon-chlorine bonds (shown in red in the figure)?



Ex. 17

Mixed Review Exercises

Give the exact values of the six trigonometric functions of each angle. If any function is not defined for the angle, say so.

1. 450° **2.** −135° **3.** 330° **4.** −240°

Write in simplest form without negative exponents.

5. $\dfrac{x^{-1} - 1}{x - x^{-1}}$ **6.** $\sqrt{50m^4}$ **7.** $\dfrac{1}{y + 2} + \dfrac{4}{y^2 - 4}$

8. $(-5a^3b)^2(2a^2b)^3$ **9.** $(x^{2/3})^{-3/4}$ **10.** $(2u - 3)(u^2 + u - 2)$

12-6 The Law of Cosines

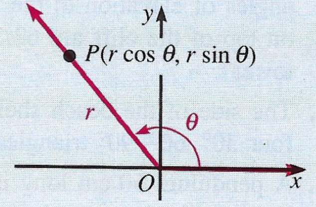
Objective To use the law of cosines to find sides and angles of triangles.

If you solve the equations

$$\sin \theta = \frac{y}{r} \quad \text{and} \quad \cos \theta = \frac{x}{r}$$

for x and y you obtain the following useful fact.

If θ is an angle in standard position and P is a point on its terminal side, then the coordinates of P are $(r \cos \theta, r \sin \theta)$, where $r = OP$.

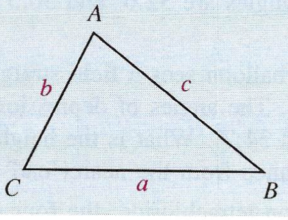


This fact is used to prove the *law of cosines*.

The Law of Cosines

In any triangle ABC,

$$c^2 = a^2 + b^2 - 2ab \cos C$$
$$b^2 = a^2 + c^2 - 2ac \cos B$$
$$a^2 = b^2 + c^2 - 2bc \cos A$$

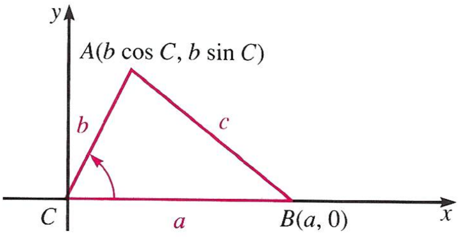


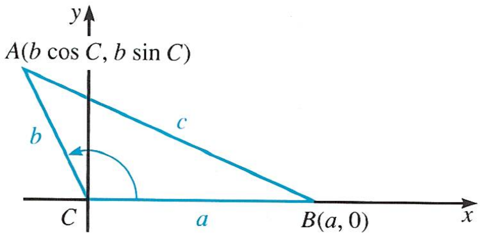
To prove the first form of the law of cosines, draw a coordinate system with $\angle C$ in standard position. (The diagrams below show an acute $\angle C$ and an obtuse $\angle C$ on such a coordinate system.) Then apply the distance formula to AB.

$$\begin{aligned} c^2 &= (b \cos C - a)^2 + (b \sin C - 0)^2 \\ &= b^2 \cos^2 C - 2ab \cos C + a^2 + b^2 \sin^2 C \\ &= a^2 + b^2(\cos^2 C + \sin^2 C) - 2ab \cos C \end{aligned}$$

But, since $\cos^2 C + \sin^2 C = 1$, the equation becomes

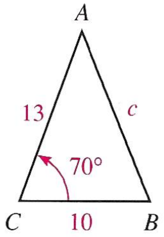
$$c^2 = a^2 + b^2 - 2ab \cos C.$$





The other two forms of the law of cosines can be proved by repeating the process just shown with $\angle A$ and $\angle B$ in turn put into standard position. Notice that if $\angle C$ is a right angle, $\cos C = 0$, and the law of cosines reduces to the Pythagorean theorem.

Example 1 In $\triangle ABC$, $a = 10$, $b = 13$, and $\angle C = 70°$. Find c to three significant digits.



Solution

$$c^2 = a^2 + b^2 - 2ab \cos C$$
$$\begin{aligned} c^2 &= 10^2 + 13^2 - 2(10)(13) \cos 70° \\ &= 100 + 169 - 260(0.3420) \\ &= 269 - 88.92 \\ &= 180.08 \end{aligned}$$
$$c = \sqrt{180.08} = 13.42$$

$\therefore$ to three significant digits, $c = 13.4$. ***Answer***

By rewriting the equation

$$c^2 = a^2 + b^2 - 2ab \cos C$$

in the form

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab},$$

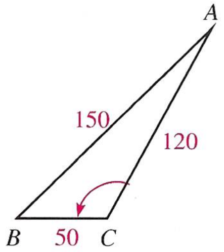
you can use the law of cosines to find the measure of an angle of a triangle when you know the lengths of the sides.

Example 2 A triangular-shaped lot has sides of length 50 m, 120 m, and 150 m. Find the largest angle of the lot to the nearest tenth of a degree.

Solution The largest angle of a triangle is opposite the longest side. Make a sketch. Let $a = 50$, $b = 120$, and $c = 150$. Substitute in this formula:

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

$$\begin{aligned} \cos C &= \frac{50^2 + 120^2 - 150^2}{2(50)(120)} \\ &= \frac{2500 + 14,400 - 22,500}{12,000} \\ &= \frac{-5600}{12,000} \\ &= -0.4667 \end{aligned}$$



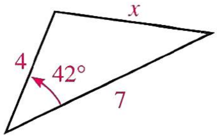
$\therefore \angle C = 117.8°$. ***Answer***

You may find it helpful to use a calculator to solve problems like the ones illustrated in Examples 1 and 2.

Oral Exercises

Use the law of cosines to give an equation involving the side or angle labeled x.

1.



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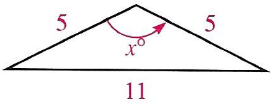

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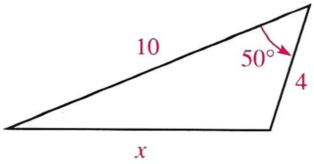

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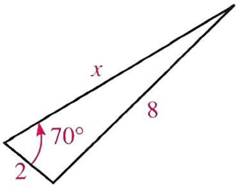

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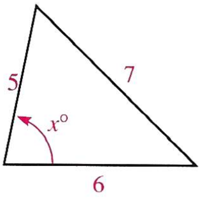

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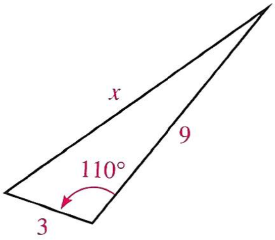


Written Exercises

Find lengths to three significant digits and the measures of the angles to the nearest tenth of a degree. You may wish to use a calculator.

In Exercises 1–12, find the indicated part of $\triangle ABC$.

A **1.** $a = 6,\ b = 7,\ \angle C = 20^\circ,\ c = \underline{\ ?\ }$

2. $b = 12,\ c = 17,\ \angle A = 74^\circ,\ a = \underline{\ ?\ }$

3. $c = 15,\ a = 13,\ \angle B = 83^\circ,\ b = \underline{\ ?\ }$

4. $b = 3,\ a = 4,\ \angle C = 40^\circ,\ c = \underline{\ ?\ }$

5. $c = 15,\ b = 30,\ \angle A = 140^\circ,\ a = \underline{\ ?\ }$

6. $a = 100,\ c = 200,\ \angle B = 150^\circ,\ b = \underline{\ ?\ }$

7. $a = 8,\ b = 10,\ c = 12,\ \angle B = \underline{\ ?\ }$

8. $a = 9,\ b = 10,\ c = 15,\ \angle C = \underline{\ ?\ }$

9. $a = 13,\ b = 30,\ c = 40$, smallest angle $= \underline{\ ?\ }$

10. $a = 30,\ b = 20,\ c = 40$, largest angle $= \underline{\ ?\ }$

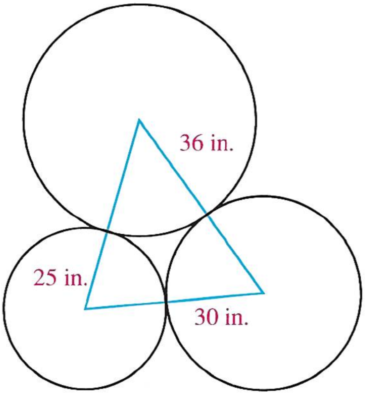
11. $a = 1.6,\ b = 0.9,\ c = 1.8$, largest angle $= \underline{\ ?\ }$

12. $a = 1.2,\ b = 2.4,\ c = 2.0$, smallest angle $= \underline{\ ?\ }$

B **13.** Given a circle O, chord $AB = 10.1$, chord $BC = 15.5$ and $\angle ABC = 26^\circ 10'$. Find the length of chord AC.

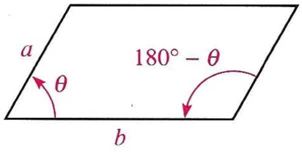
14. A parallelogram has sides 6 cm and 8 cm and a 65° angle. Find the lengths of the diagonals. (Recall that adjacent angles of a parallelogram are supplementary.)

15. Three circles with radii 25 in., 36 in., and 30 in. are externally tangent to each other. Find the angles of the triangle formed by joining their centers.



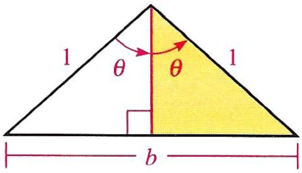
16. Find the lengths of the sides of a parallelogram whose diagonals intersect at a 35° angle and have lengths 6 and 10. (Recall that the diagonals of a parallelogram bisect each other.)

17. In a parallelogram having sides of lengths a and b, let the diagonals have lengths d_1 and d_2. Show that $d_1^2 + d_2^2 = 2(a^2 + b^2)$. (*Hint:* Apply the law of cosines twice and use the fact that $\cos(180° - \theta) = -\cos\theta$.)



C 18. In $\triangle ABC$, $a = 2$, $b = 4$, and $c = 3$. Find the length of the median from A to $\overline{BC}$.

19. Prove that $\cos 2\theta = 1 - 2\sin^2\theta$ for $0° < \theta < 90°$ as follows. First, using the triangle shown, express b^2 in two ways: (1) by using the law of cosines and (2) by first finding $\frac{b}{2}$ from the shaded right triangle. Then set the two expressions equal.



Problems

Find lengths to three significant digits and measures of angles to the nearest tenth of a degree. You may wish to use a calculator.

A 1. A ranger in an observation tower can sight the north end of a lake 15 km away and the south end of the same lake 19 km away. The angle between these two lines of sight is 104°. How long is the lake?

2. Two planes leave an airport at the same time, one flying due west at 500 km/h and the other flying due southeast at 300 km/h. What is the distance between the planes two hours later?

3. A triangular-shaped lot of land has sides of length 130 m, 150 m, and 80 m. What are the measures of the angles?

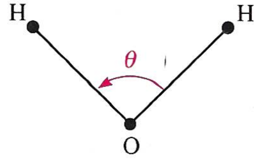
4. Two streets meet at an angle of 52°. If a triangular lot has frontages of 60 m and 65 m on the two streets, what is the perimeter of the lot?

5. Newtown is 8 mi east of Oldtown and Littleton is 10 mi northwest of Oldtown. How far is Newtown from Littleton?

6. An oil tanker and a cruise ship leave port at the same time and travel straight-line courses at 10 mi/h and 25 mi/h, respectively. Two hours later they are 40 mi apart. What is the angle between their courses?

7. A baseball diamond is a square 90 ft on a side. The pitcher's mound is 60.5 feet from home plate. How far is it from the mound to first base?

8. A water molecule consists of two hydrogen atoms and one oxygen atom joined as in the diagram. The distance from the nucleus of each hydrogen atom to the nucleus of the oxygen atom is 9.58×10^{-9} cm, and the bond angle θ is 104.8°. How far are the nuclei of the hydrogen atoms from each other?



B **9.** A large park in the shape of a parallelogram has diagonal paths that meet at a 60° angle. If the diagonal paths are 12 km and 20 km long, find the perimeter and the area of the park.

10. A flagpole 4 m tall stands on a sloping roof. A support wire 5 m long joins the top of the pole to a point on the roof 6 m up from the bottom of the pole. At what angle is the roof inclined to the horizontal?

11. A vertical pole 20 m tall standing on a 15° slope is braced by two cables extending from the top of the pole to two points on the ground, 30 m up the slope and 30 m down the slope. How long are the cables?

C **12.** The measures of two sides of a parallelogram are 50 cm and 80 cm, and one diagonal is 90 cm long. How long is the other diagonal?

Computer Exercises

For students with some programming experience.

1. Given a right triangle ABC with $\angle C = 90°$ and the lengths of two sides, write a program that will find the length of the other side and the degree measures of the two acute angles. (*Hint:* You can find the measure of the acute angles by using a ratio of the lengths of the legs as the argument for the computer's built-in inverse tangent function ATN. This function gives an angle measure in radians (see Lesson 13-1). To convert radian measure R to degree measure D you can use the formula D = R∗180/3.14159.)

2. Run the program in Exercise 1 to find the length of the remaining side and the measures of the remaining angles in each right triangle.

a. $a = 52$, $b = 38$ **b.** $a = 29.6$, $c = 47.4$ **c.** $b = 5.93$, $c = 9.07$

3. Given a right triangle ABC with $\angle C = 90°$ and the length of one side and the measure of one acute angle, write a program that will find the lengths of the other two sides and the measure of the third angle. (*Hint:* Remember that the computer's functions SIN(X), COS(X), and TAN(X) require an argument X in radians. To convert degree measure D to radian measure R you can use the formula R = D∗3.14159/180.)

4. Run the program in Exercise 3 to find the lengths of the remaining sides and the measure of the remaining angle in each triangle.

a. $\angle A = 32°$, $a = 37$ **b.** $\angle B = 15°$, $b = 17.6$ **c.** $\angle B = 19.8°$, $c = 9.89$

5. Given the degree measure of two angles and the lengths of two sides of a triangle, write a program that will find the degree measure of the third angle and the length of the other side.

6. Run the program in Exercise 5 to find the measures of the remaining sides and angle in each triangle.

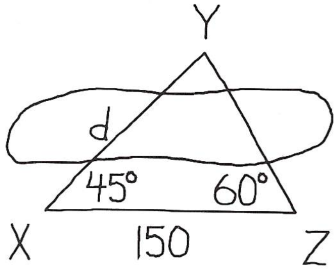
a. $\angle A = 68°$, $\angle B = 72°$, $c = 13.91$, $a = 20.06$
b. $\angle A = 51.3°$, $\angle B = 102.8°$, $c = 27.05$, $b = 60.39$
c. $\angle B = 115.6°$, $\angle C = 12.4°$, $a = 7.385$, $c = 2.012$

Reading Algebra / Making a Sketch

Read the following problem and try to visualize how the points, angles, and distances are related.

> Two points, X and Y, are separated by a swamp. To find the distance between them, Kelly walks 150 m from X to another point, Z. She estimates $\angle ZXY$ to be 45° and $\angle XZY$ to be 60°. Find the approximate distance from X to Y.

Can you picture the relative positions of X, Y, and Z? Many problems dealing with distance, angles, and geometric shapes are difficult to think about if you have only words to look at. Making a sketch as you read can help you understand the given information and see what is being asked for. For example, a sketch like the one below shows all the essential information in the problem stated above.



Notice that your sketch for a problem can be useful even if it is not to scale. For example, in the sketch above, there is no need to draw the angles exactly to scale, though we have shown them both as acute, with the larger one at Z.

Most diagrams in your textbook are drawn to scale. When a diagram is drawn for your book, the artist begins with complete information about dimensions. But you usually won't know the exact proportions of the shapes in your sketch until you have solved the problem. So don't wait until your solution is complete to make a drawing. Start right away with a rough sketch; you can improve it later as you discover more information.

Exercises

For each problem listed below, draw a sketch showing the essential information, labeling segments and angles with their given measures, and using variables of your choice to represent unknowns.

1. Problem 5, page 589

2. Problem 10, page 590

3. Problem 1, page 595

4. Problem 12, page 596

12-7 The Law of Sines

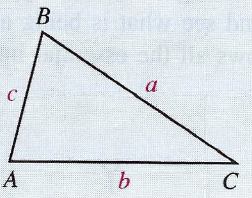
Objective To use the law of sines to find sides and angles of triangles.

The *law of sines,* like the law of cosines, allows you to use given information about three measurements of a triangle to determine the other measurements. For example, if you know the lengths of two sides and the measure of the included angle, you can find the length of the third side by using the law of cosines. But if the given angle is not the included angle, the law of cosines cannot be used and you need to use the law of sines.

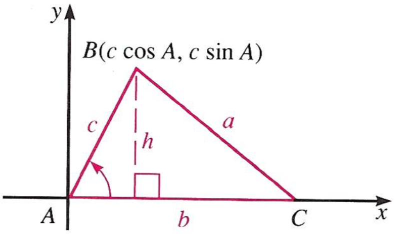
The Law of Sines

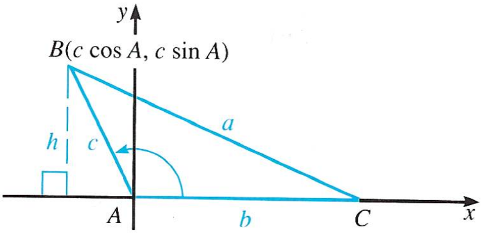
In any triangle ABC,

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$



To prove the law of sines, you first find a formula for the area, K, of $\triangle ABC$. Draw a coordinate system with $\angle A$ in standard position. ($\angle A$ may be acute or obtuse.) Let h be the height of the triangle measured from B.





Since $K = \frac{1}{2}$(base)(height), you can write

$$K = \frac{1}{2}bh.$$

Since the y-coordinate of B equals h, you know $h = c \sin A$. Therefore,

$$K = \frac{1}{2}bc \sin A.$$

You can find two other formulas for the area K of $\triangle ABC$ by replacing $\angle A$ in turn with $\angle B$ and $\angle C$:

$$K = \frac{1}{2}ac \sin B \qquad K = \frac{1}{2}ab \sin C$$

Therefore, by substitution,

$$\frac{1}{2}bc \sin A = \frac{1}{2}ac \sin B = \frac{1}{2}ab \sin C.$$

By dividing each expression by $\frac{1}{2}abc$, you obtain the law of sines shown above.

Example 1 In $\triangle ABC$, $\angle A = 40°$ and $a = 15$.

a. Find $\angle B$ if $b = 20$.

b. Find $\angle B$ if $b = 8$.

Solution Use the formula $\frac{\sin B}{b} = \frac{\sin A}{a}$.

a.
$$\frac{\sin B}{20} = \frac{\sin 40°}{15}$$
$$\sin B = \frac{20 \sin 40°}{15}$$
$$= \frac{20(0.6428)}{15}$$
$$= 0.8571$$
$\angle B = 59°$ or $\angle B = 121°$

Since $40° + 59° = 99°$ and $40° + 121° = 161°$, and both 99° and 161° are less than 180°, there are two solutions.

$\therefore \angle B = 59°$ or
$\angle B = 121°$. ***Answer***

b.
$$\frac{\sin B}{8} = \frac{\sin 40°}{15}$$
$$\sin B = \frac{8 \sin 40°}{15}$$
$$= \frac{8(0.6428)}{15}$$
$$= 0.3428$$
$\angle B = 20°$ or $\angle B = 160°$

160° is not a solution because $\angle A + \angle B = 40° + 160° = 200°$, which is greater than 180°.

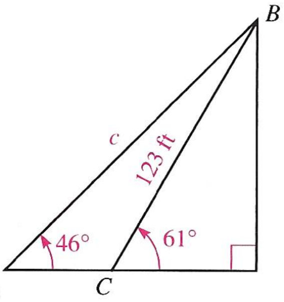
$\therefore \angle B = 20°$. ***Answer***

If you are given two sides of a triangle and the angle opposite one of the sides, there may be two solutions, one solution, or no solution. You'll learn why in Lesson 12-8.

Example 2 A 123 ft support wire for a transmitting tower makes an angle of 61° with the ground. This wire is to be replaced by a new wire whose angle with the ground is 46°. How long will the new wire be?

Solution Let $\angle A = 46°$, and let $c =$ the length of the new wire.
$\angle ACB = 180° - 61° = 119°$, and $a = 123$ (ft).

$$\frac{\sin A}{a} = \frac{\sin ACB}{c}$$
$$\frac{\sin 46°}{123} = \frac{\sin 119°}{c}$$
$$c = \frac{123 \sin 119°}{\sin 46°}$$
$$= \frac{123(0.8746)}{0.7193}$$
$$= 149.6$$



$\therefore$ the new wire will be 149.6 ft long. ***Answer***

Oral Exercises

Given triangle ABC with sides a, b, and c, use the law of sines to find an expression equivalent to the given one.

1. $\frac{b}{a}$

2. $\frac{a}{\sin A}$

3. $\frac{\sin A}{\sin B}$

4. $\frac{a \sin B}{b \sin A}$

5. $\frac{a \sin B}{b}$

6. $\frac{\cos A}{\sin A}$ if $\angle C = 90°$

Written Exercises

Find the indicated part of $\triangle ABC$ to three significant digits or to the nearest tenth of a degree. If there are two solutions, give both. You may wish to use a calculator.

A **1.** $a = 14$, $\angle A = 25°$, $\angle B = 75°$, $b = \underline{\ ?\ }$

2. $c = 12$, $\angle A = 42°$, $\angle C = 69°$, $a = \underline{\ ?\ }$

3. $b = 3.40$, $\angle A = 110°$, $\angle C = 50°$, $a = \underline{\ ?\ }$

4. $a = 2.60$, $\angle B = 60°$, $\angle C = 100°$, $c = \underline{\ ?\ }$

5. $c = 35$, $\angle A = 38°$, $\angle C = 102°$, $b = \underline{\ ?\ }$

6. $b = 130$, $\angle B = 95°$, $\angle C = 35°$, $a = \underline{\ ?\ }$

7. $a = 4$, $b = 3$, $\angle A = 40°$, $\angle B = \underline{\ ?\ }$

8. $a = 4.5$, $b = 6.0$, $\angle B = 35°$, $\angle A = \underline{\ ?\ }$

9. $a = 4.0$, $c = 6.4$, $\angle C = 125°$, $\angle B = \underline{\ ?\ }$

10. $a = 18$, $b = 12$, $\angle A = 110°$, $\angle C = \underline{\ ?\ }$

11. $a = 5$, $c = 7$, $\angle A = 42°$, $\angle C = \underline{\ ?\ }$

12. $b = 15$, $c = 11$, $\angle C = 40°$, $\angle B = \underline{\ ?\ }$

In Exercises 13–18, find the exact value of $\frac{a}{b}$ in $\triangle ABC$ without using tables or calculators. (Recall that $\sin^2 \theta + \cos^2 \theta = 1$.)

Sample $\sin A = \frac{3}{4}$, $\cos B = \frac{2}{3}$

Solution The law of sines can be written in the form $\frac{a}{b} = \frac{\sin A}{\sin B}$. You need to find $\sin B$. Since $\sin^2 B + \cos^2 B = 1$, you have $\sin^2 B = 1 - \cos^2 B = 1 - \left(\frac{2}{3}\right)^2 = \frac{5}{9}$. Therefore, $\sin B = \frac{\sqrt{5}}{3}$.

By substitution, $\frac{a}{b} = \frac{\sin A}{\sin B} = \frac{\frac{3}{4}}{\frac{\sqrt{5}}{3}} = \frac{9}{4\sqrt{5}} = \frac{9\sqrt{5}}{20}$.

13. $\sin A = \frac{2}{3}$, $\cos B = \frac{4}{5}$

14. $\cos A = \frac{3}{5}$, $\sin B = \frac{4}{5}$

15. $\cos A = \frac{5}{13}$, $\cos B = \frac{3}{5}$

16. $\cos A = \frac{8}{17}$, $\cos B = \frac{15}{17}$

17. $\cos A = \frac{\sqrt{3}}{2}$, $\cos B = \frac{\sqrt{2}}{2}$

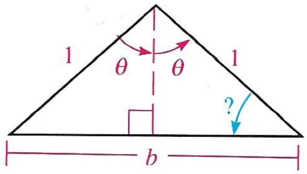
18. $\cos A = \frac{1}{2}$, $\tan B = 1$

Show that the following formulas are true for any triangle ABC.

B 19. $\dfrac{\sin A + \sin B}{\sin B} = \dfrac{a + b}{b}$

20. $\dfrac{\sin A - \sin B}{\sin B} = \dfrac{a - b}{b}$

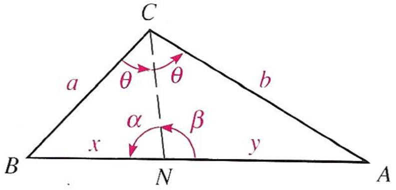
21. Prove the formula $\sin 2\theta = 2 \sin \theta \cos \theta$ for $0° < \theta < 90°$. Use the law of sines and the fact that $\sin (90° - \theta) = \cos \theta$.



22. Use the law of sines to show that the bisector of an angle of a triangle divides the opposite side in the same ratio as the two adjacent sides; that is,

$$\frac{a}{b} = \frac{x}{y}.$$

(*Hint:* Use the law of sines in $\triangle BNC$ and $\triangle ANC$ and the fact that $\sin (180° - B) = \sin B$.)



Problems

Give answers to three significant digits. You may wish to use a calculator.

A 1. Two angles of a triangle measure 32° and 53°. The longest side is 55 cm. Find the length of the shortest side.

2. Two angles of a triangle measure 75° and 51°. The side opposite the 75° angle is 25 in. How long is the shortest side?

3. How long is the base of an isosceles triangle if each leg is 27 cm and each base angle measures 23°?

4. A college football pennant is in the shape of an isosceles triangle. The base is 16 in. long. The sides meet at an angle of 35°. How long are the sides?

5. A fire is sighted from two ranger stations that are 5000 m apart. The angles of observation to the fire measure 52° from one station and 41° from the other station. Find the distance along the line of sight to the fire from the closer of the two stations.

6. Two surveyors are on opposite sides of a swamp. To find the distance between them, one surveyor locates a point T that is 180 m from her location at point P. The angles of sight from T to the other surveyor's position, R, measure 72° for $\angle RPT$ and 63° for $\angle PTR$. How far apart are the surveyors?

7. Two markers are located at points A and B on opposite sides of a lake. To find the distance between the markers, a surveyor laid off a base line, $\overline{AC}$, 25 m long and found that $\angle BAC = 85°$ and $\angle BCA = 66°$. Find AB.

8. From a hang glider approaching a 5000 ft clearing the angles of depression of the opposite ends of the field measure 24° and 30°. How far is the hang glider from the nearer end of the field?

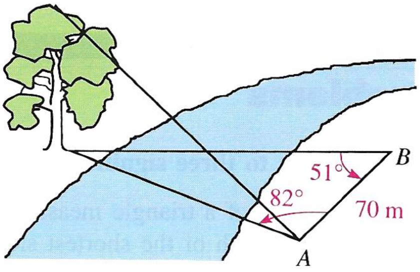
B 9. A loading ramp 5 m long makes a 25° angle with the level ground beneath it. The ramp is replaced by another ramp 15 m long. Find the angle that the new ramp makes with the ground.

10. From the top of an office building 72 ft high, the angle of elevation to the top of an apartment building across the street is 31°. From the base of the office building, the angle of elevation is 46°. How tall is the apartment building?

11. The captain of a freighter 6 km from the nearer of two unloading docks on the shore finds that the angle between the lines of sight to the two docks is 35°. If the docks are 10 km apart, how far is the tanker from the farther dock?

12. Two wires bracing a transmission tower are attached to the same location on the ground. One wire is attached to the tower 10 m above the other. The longer wire is 30 m long and the angle of inclination of the shorter wire is 53°. To the nearest tenth of a degree, what angle does the longer wire make with the ground?

C 13. To find the height of a tree across a river, Carlos laid off a base line 70 m long and measured the angles shown. He found the angle of elevation of the top of the tree from A to be 10°. How tall is the tree?



Mixed Review Exercises

Give the following function values to four significant digits.

1. cos 78.2°
2. tan (−43°10′)
3. sin 213.7°
4. csc 305°45′
5. cot (−101.9°)
6. sec 488°34′

Write an equation for each figure described.

7. The circle with center (−1, 4) and radius 3.
8. The line through (−3, 5) and (2, 0).
9. The parabola with focus (0, −2) and directrix $y = 2$.

12-8 Solving General Triangles

Objective To solve any given triangle.

The problem of solving triangles can be divided into four cases:

SSS: Given three sides.

SAS: Given two sides and the included angle.

SSA: Given two sides and the angle opposite one of them.

ASA and **AAS:** Given two angles and one side.

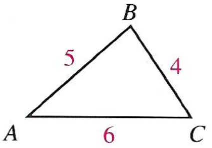
You may remember that SSS, SAS, ASA, and AAS are used in geometry to prove triangles congruent. However, knowing two sides and a nonincluded angle (the SSA case) is not enough to prove two triangles congruent. You will see later in this lesson that when you try to solve a triangle given these measurements, they may not determine a triangle.

In the examples that follow, lengths will be found to three significant digits and angle measures to the nearest tenth of a degree.

Example 1 (SSS case) Solve $\triangle ABC$ if $a = 4$, $b = 6$, and $c = 5$.

Solution Use the law of cosines to find one of the angles, say $\angle A$.

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$



$$\cos A = \frac{6^2 + 5^2 - 4^2}{2 \cdot 6 \cdot 5}$$

$$= \frac{36 + 25 - 16}{60}$$

$$= \frac{45}{60} = 0.7500$$

$\therefore \angle A = 41.4°$

To find another angle, it is usually easier to use the law of sines.

$$\frac{\sin C}{c} = \frac{\sin A}{a}$$

$$\sin C = \frac{c \sin A}{a}$$

$$\sin C = \frac{5 \sin 41.4°}{4} = \frac{5(0.6613)}{4} = 0.8266$$

Since $\angle C$ is not the largest angle, it must be acute.

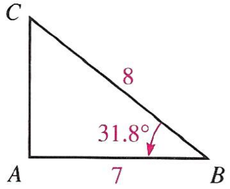
$\therefore \angle C = 55.8°$

Since $\angle A + \angle B + \angle C = 180°$, you know $\angle B = 180° - \angle A - \angle C$. So by substitution, $\angle B = 180° - 41.4° - 55.8° = 82.8°$.

$\therefore \angle A = 41.4°$, $\angle B = 82.8°$, and $\angle C = 55.8°$. ***Answer***

Example 2 (SAS case) Solve $\triangle ABC$ if $a = 8$, $c = 7$, and $\angle B = 31.8°$.

Solution Use the law of cosines to find the third side, b.



$$\begin{aligned} b^2 &= a^2 + c^2 - 2ac \cos B \\ &= 8^2 + 7^2 - 2(8)(7) \cos 31.8° \\ &= 64 + 49 - 112(0.8499) \\ &= 17.8 \end{aligned}$$

$$b = \sqrt{17.8} = 4.2$$

Use the law of sines to find the measure of $\angle C$, the smaller of the two remaining angles.

$$\frac{\sin C}{c} = \frac{\sin B}{b}$$

$$\begin{aligned} \sin C &= \frac{c \sin B}{b} \\ &= \frac{7 \sin 31.8°}{4.2} \\ &= \frac{7(0.5270)}{4.2} = 0.8783 \end{aligned}$$

Since $\angle C$ is not the largest angle, it must be acute.

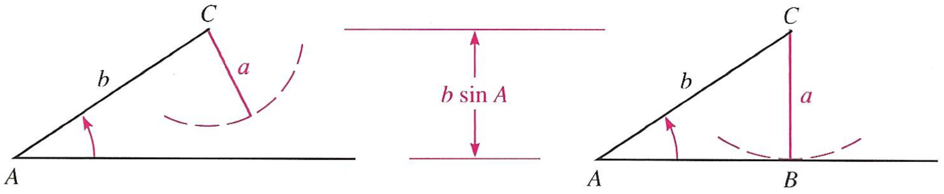
$\therefore \angle C = 61.4°$

$$\begin{aligned} \angle A &= 180° - 31.8° - 61.4° \\ &= 86.8° \end{aligned}$$

$\therefore \angle A = 86.8°$, $b = 4.2$, and $\angle C = 61.4°$. ***Answer***

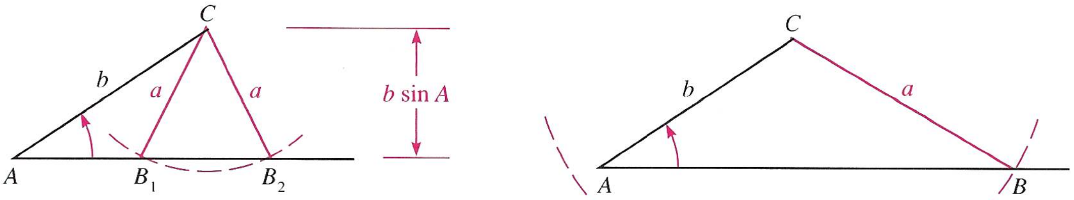
The SSA case is often called the *ambiguous case* because there are six possible outcomes, four if $\angle A$ is acute and two if $\angle A$ is right or obtuse. These possibilities are illustrated below and on the opposite page.

$\angle A < 90°$



$a < b \sin A$
No solution

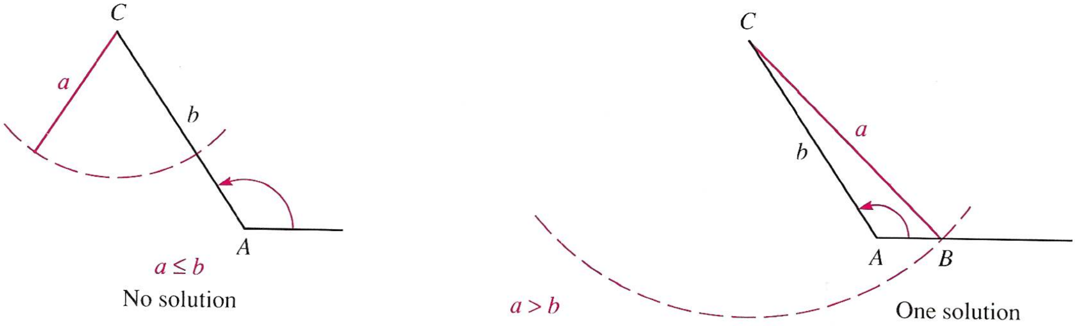
$a = b \sin A$
One solution



$b \sin A < a < b$
Two solutions

$a \geq b$
One solution

$\angle A \geq 90°$



$a \leq b$
No solution

$a > b$
One solution

Example 3 (SSA case) Solve $\triangle ABC$ if $b = 22$, $c = 30$, and $\angle B = 30°$.

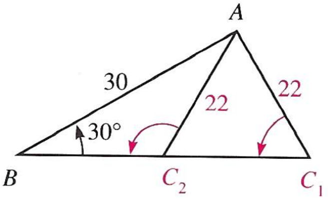
Solution

$$\frac{\sin C}{c} = \frac{\sin B}{b}$$

$$\sin C = \frac{c \sin B}{b}$$

$$= \frac{30 \sin 30°}{22}$$

$$= \frac{30(0.5000)}{22} = 0.6818$$



$\angle C = 43.0°$
$\angle A = 180° - 30° - 43.0° = 107.0°$

$$a = \frac{b \sin A}{\sin B}$$

$$= \frac{22 \sin 107.0°}{\sin 30°}$$

$$= \frac{22(0.9563)}{0.5000}$$

$a = 42.1$

or

$\angle C = 180° - 43.0° = 137.0°$
$\angle A = 180° - 30° - 137.0° = 13.0°$

$$a = \frac{b \sin A}{\sin b}$$

$$= \frac{22 \sin 13.0°}{\sin 30°}$$

$$= \frac{22(0.2250)}{0.5000}$$

$a = 9.90$

$\therefore \angle A = 107.0°$, $\angle C = 43.0°$, and $a = 42.1$
or $\angle A = 13.0°$, $\angle C = 137.0°$, and $a = 9.90$ ***Answer***

Example 4 (AAS case) Solve $\triangle ABC$ if $a = 40$, $\angle A = 45°$, and $\angle C = 55°$.

Solution

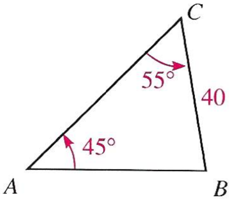
$$\angle B = 180° - \angle A - \angle C = 180° - 45° - 55° = 80°$$

$$b = \frac{a \sin B}{\sin A} = \frac{40 \sin 80°}{\sin 45°} = \frac{40(0.9848)}{0.7071}$$

$b = 55.7$

$$c = \frac{a \sin C}{\sin A} = \frac{40 \sin 55°}{\sin 45°} = \frac{40(0.8192)}{0.7071}$$

$c = 46.3$



$\therefore b = 55.7$, $c = 46.3$, and $\angle B = 80°$. ***Answer***

Oral Exercises

Given the following measurements, tell which law you would use first to solve the triangle.

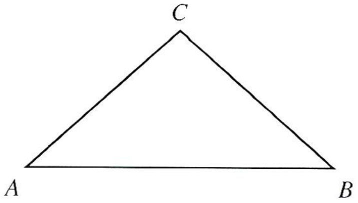
1. SSS **2.** SAS **3.** SSA **4.** AAS **5.** ASA

Outline a strategy for solving each triangle.

Sample $a = 20$, $c = 30$, $\angle B = 40°$

Solution

1. Use the law of cosines to find b.
2. Use the law of sines to find $\angle A$.
3. Use $\angle C = 180° - \angle A - \angle B$ to find $\angle C$.



6. $a = 17$, $\angle B = 20°$, $\angle C = 60°$

7. $a = 8$, $c = 13$, $\angle B = 150°$

8. $a = 5$, $b = 7$, $c = 11$

9. $b = 15$, $\angle B = 100°$, $\angle C = 25°$

10. $b = 10$, $c = 13$, $\angle A = 120°$

11. $a = 10$, $b = 25$, $c = 30$

12. $a = 30$, $b = 20$, $\angle A = 130°$

13. $b = 20$, $c = 18$, $\angle B = 120°$

14. $a = 9$, $b = 10$, $\angle B = 40°$

15. $a = 12$, $b = 11$, $\angle A = 35°$

16. $b = 14$, $c = 18$, $\angle B = 50°$

17. $b = 15$, $c = 14.5$, $\angle C = 70°$

18. In $\triangle ABC$, if $\angle A$ is an obtuse angle, what can you conclude about $\angle C$?

19. In $\triangle ABC$, if b is the longest side, what can you conclude about $\angle A$?

20. Explain why the SSA case is called the ambiguous case.

Written Exercises

Give lengths to three significant digits and angle measures to the nearest tenth of a degree. You may wish to use a calculator.

A **1–12.** Solve the triangles given in Oral Exercises 6–17. If there are two solutions, find both. If there are no solutions, say so.

In Exercises 13 and 14, recall that a median is the segment from a vertex to the midpoint of the opposite side.

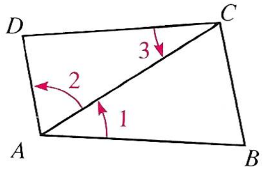
B **13.** In $\triangle ABC$, $a = 4$, $b = 5$, and $\angle C = 110°$. Find the length of the median to the longest side.

14. In $\triangle ABC$, $c = 10$ and $\angle A = \angle B = 40°$. Find the length of the median to $\overline{AC}$.

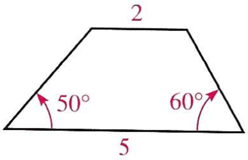
Solve Exercises 15 and 16 by using both the law of sines and the law of cosines. You may wish to use a calculator.

15. In quadrilateral $ABCD$, $\angle 1 = 30°$, $\angle 2 = 80°$, $\angle B = 80°$, $AD = 8$, and $AB = 10$. Find the length of $\overline{CD}$.

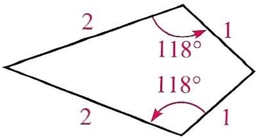
16. In quadrilateral $ABCD$, $\angle B = 110°$, $\angle D = 75°$, $\angle 3 = 35°$, $AB = 5$, and $BC = 6$. Find the length of $\overline{CD}$.



17. Find the lengths of the diagonals of the trapezoid shown below.



18. Find the lengths of the diagonals of the quadrilateral shown below.



C **19.** The angles of a triangle measure 57°, 60°, and 63°. The longest side is 5 units longer than the shortest side. Find the lengths of the three sides.

20. One angle of a triangle has measure 65°. The side opposite this angle has length 10. If the area of the triangle is 20, find the perimeter.

Problems

Give lengths to three significant digits and angle measures to the nearest tenth of a degree. You may wish to use a calculator.

A **1.** Two planes leave an airport at the same time, one flying at 300 km/h and the other at 420 km/h. The angle between their flight paths is 75°. After three hours, how far apart are they?

2. Jan is flying a plane on a triangular course at 320 mi/h. She flies due east for two hours and then turns right through a 65° angle. How long after turning will she be exactly southeast of where she started?

3. Two cables of length 300 m and 270 m extend from the top of a television antenna to the level ground on opposite sides of the antenna. The longer cable makes an angle of 48° with the ground. Find the acute angle that the shorter cable makes with the ground and the distance between the cables along the ground.

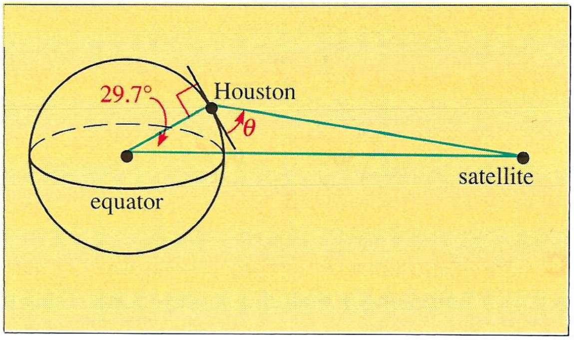
4. A vertical television mast is mounted on the roof of a building. From a point 750 ft from the base of the building, the angles of elevation to the bottom and top of the mast measure 34° and 50° respectively. How tall is the mast?

5. Jim is flying a plane from Midville to Vista, a distance of 500 km. Because of a thunderstorm, he had to fly 17.5° off course for 300 km. How far is he now from Vista, and through what angle should he turn to fly directly there?

6. A monument consists of a flagpole 15 m tall standing on a mound in the shape of a cone with vertex angle 140°. How long a shadow does the pole cast on the cone when the angle of elevation of the sun is 62°?

B 7. From the top of a building 10 m tall, the angle of elevation to the top of a flagpole is 11°. At the base of the building, the angle of elevation to the top of the flagpole is 39°. Find the height of the flagpole.

8. A communication satellite is in orbit 35,800 km above the equator. It completes one orbit every 24 hours, so that from Earth it appears to be stationary above a point on the equator. If this point has the same longitude as Houston, find the measure of θ, the satellite's angle of elevation from Houston. The latitude of Houston is 29.7°N. The radius of Earth is 6400 km.



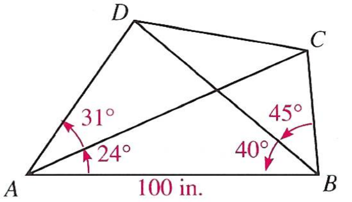
9. In Problem 8, what is the greatest latitude from which a signal can travel to the satellite in a straight line?

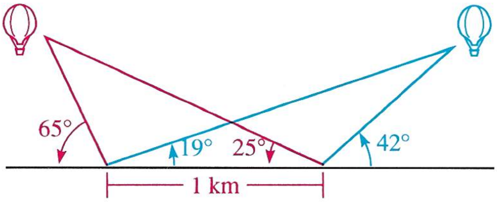
10. From the top of an observation post that is 90 m high, a ranger sights a campsite at an angle of depression of 10°. Turning in a different direction, the ranger sees another campsite at an angle of depression of 13°. The angle between these two lines of sight is 35°. How far apart are the campsites?

C 11. In quadrilateral $ABCD$, $AB = 3$, $BC = 4$, $CD = 5$, and $DA = 6$. The length of diagonal $\overline{BD}$ is 7. Find the length of the other diagonal.

12. From Base Camp, located 2000 m above sea level, the angle of elevation of Camp A is 25° and the angle of elevation of the summit of Mount Snow is 41°. From Camp A the angle of elevation of the summit is 53°. If Camp A is 3 km closer to the summit (air distance) than Base Camp is, how high is the summit above sea level?

13. In quadrilateral $ABCD$ shown at the left below, find the length of $\overline{CD}$.





14. Two balloons are moored directly over a straight, level road. The diagram on the right above shows the angle of elevation of the balloons from two observers on the road one kilometer apart. How far apart are the balloons? Which balloon is higher, and by how many meters?

12-9 Areas of Triangles

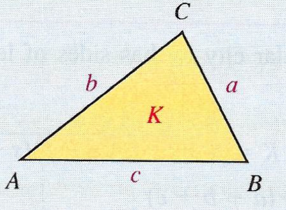
Objective To apply triangle area formulas.

You know that the area of a triangle can be found by using the formula

$$K = \frac{1}{2}(\text{base})(\text{height}).$$

Here are some other area formulas for triangles.

The area K of $\triangle ABC$ is given by each of the formulas listed below.



$K = \frac{1}{2}bc \sin A$ $\quad K = \frac{1}{2}ac \sin B$ $\quad K = \frac{1}{2}ab \sin C$

$K = \frac{1}{2}a^2 \frac{\sin B \sin C}{\sin A}$ $\quad K = \frac{1}{2}b^2 \frac{\sin A \sin C}{\sin B}$ $\quad K = \frac{1}{2}c^2 \frac{\sin A \sin B}{\sin C}$

$K = \sqrt{s(s-a)(s-b)(s-c)}$, where $s = \frac{1}{2}(a + b + c)$

The first three formulas are useful for finding areas of triangles in the SAS case. They were derived in Lesson 12-7. The formulas in the second row above are useful in the AAS case. You can prove them using the law of sines in the form $b = \frac{a \sin B}{\sin A}$. By substituting $\frac{a \sin B}{\sin A}$ for b in $K = \frac{1}{2}ab \sin C$, the equation becomes $K = \frac{1}{2}a^2 \frac{\sin B \sin C}{\sin A}$. The other formulas on the second row are proved similarly. The formula on the third row is called **Hero's formula** and is used to find areas in the SSS case. Its proof is outlined in Exercises 17–19 on pages 599–600.

Example 1 (SAS case) Find the area of $\triangle ABC$ if $b = 32$, $c = 27$, and $\angle A = 108°$.

Solution $K = \frac{1}{2}bc \sin A = \frac{1}{2}(32)(27) \sin 108°$

$= 432\ (0.9511)$

$= 411$

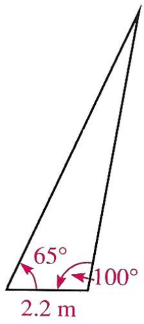
$\therefore$ the area is 411 square units. ***Answer***

Example 2 (ASA case) Find the area of the triangle shown at the right.

Solution Let $\angle A = 100°$ and $\angle B = 65°$.
Then $\angle C = 180° - 100° - 65° = 15°$.

Use the formula $K = \frac{1}{2}c^2 \frac{\sin A \sin B}{\sin C}$.

$$\begin{aligned} K &= \frac{1}{2}(2.2)^2 \frac{\sin 100° \sin 65°}{\sin 15°} \\ &= 2.42 \frac{(0.9848)(0.9063)}{(0.2588)} \\ &= 8.35 \end{aligned}$$



$\therefore$ the area is 8.35 m^2. ***Answer***

Example 3 (SSS case) A triangular city lot has sides of lengths 50 ft, 60 ft, and 80 ft. Find its area.

Solution Use Hero's formula: $K = \sqrt{s(s-a)(s-b)(s-c)}$, where

$$\begin{aligned} s &= \frac{1}{2}(a + b + c) \\ &= \frac{1}{2}(50 + 60 + 80) = 95. \end{aligned}$$

Then: $s - a = 95 - 50 = 45$
$s - b = 95 - 60 = 35$
$s - c = 95 - 80 = 15$

Now substitute: $K = \sqrt{s(s-a)(s-b)(s-c)}$
$= \sqrt{95(45)(35)(15)} = 75\sqrt{399} = 1498$

$\therefore$ to three significant digits, the area is 1500 ft^2. ***Answer***

In the ambiguous case (SSA), the triangle must be partially solved before the area, if any, can be found.

Example 4 (SSA case) Find the area of $\triangle ABC$ if $b = 22$, $c = 30$ and $\angle B = 30°$.

Solution In Example 3, page 593, two triangles fit the given data, one with $\angle A = 107.0°$ and the other with $\angle A = 13.0°$. Therefore:

$\angle A = 107.0°$ or $\angle A = 13.0°$

$$\begin{aligned} K &= \frac{1}{2}bc \sin A \\ &= \frac{1}{2}(22)(30) \sin 107.0° \\ &= 330(0.9563) \\ &= 316 \end{aligned}$$

$$\begin{aligned} K &= \frac{1}{2}bc \sin A \\ &= \frac{1}{2}(22)(30) \sin 13.0° \\ &= 330(0.2250) \\ &= 74.3 \end{aligned}$$

$\therefore$ the area is 316 square units or 74.3 square units. ***Answer***

Oral Exercises

Three parts of $\triangle ABC$ are given. Tell which formula you would use to find the area of the triangle.

1. $\angle A = 25°$, $\angle B = 50°$, $b = 30$

2. $a = 18$, $b = 10$, $\angle C = 45°$

3. $a = 6$, $c = 14$, $\angle B = 62°$

4. $\angle A = 70°$, $b = 15$, $c = 36$

5. $a = 12$, $b = 8$, $c = 12$

6. $\angle B = 20°$, $\angle C = 115°$, $c = 16$

7. $c = 15$, $b = 22$, $\angle A = 150°$

8. $a = 7$, $b = 12$, $c = 8$

9. Suppose $a = 12$, $b = 11$, and $\angle B = 60°$. Explain why this is an example of the ambiguous case.

Written Exercises

Give lengths to three significant digits and angle measures to the nearest tenth of a degree. You may wish to use a calculator.

A **1–8.** Find the areas of each triangle described in Oral Exercises 1–8.

9. Find the area of a parallelogram that has a 45° angle and sides with lengths 10 and 18.

10. Find the area of a rhombus that has perimeter 48 and an angle of 55°.

B **11.** The area of $\triangle ABC$ is 36 square units. If $\angle B = 30°$ and $c = 8$, find the value of a.

12. The area of $\triangle ABC$ is 20 square units. If $\angle A = 130°$ and $b = 6$, find the value of c.

13. A triangle has area 48 cm^2 and its shorter sides have lengths 9 cm and 12 cm. Find the largest angle of the triangle.

14. The angles of a triangle are 25°, 45°, and 110°. What is the length of the longest side if the area of the triangle is 75 square units?

15. Find the area of a regular octagon whose sides are 10 cm long.

16. Find the area of a regular octagon inscribed in a unit circle.

Exercises 17–19 make up a proof of Hero's formula.
Exercises 17 and 18 are used in Exercise 19 on the following page.

17. Show that $K^2 = \frac{1}{4}a^2b^2 \sin^2 C$.

18. Recall that $s = \frac{1}{2}(a + b + c)$, so that $2s = a + b + c$.

Therefore, $2(s - a) = 2s - 2a = a + b + c - 2a = -a + b + c$.
Show that $2(s - b) = a - b + c$ and $2(s - c) = a + b - c$.

C **19.** Justify each of the statements from (a)–(i) below. Use the results of Exercises 17 and 18 where necessary.

a. $a^2 + b^2 - c^2 = 2ab \cos C$

b. $(a^2 + b^2 - c^2)^2 = 4a^2b^2 \cos^2 C = 4a^2b^2(1 - \sin^2 C)$

c. $(a^2 + b^2 - c^2)^2 = 4a^2b^2 - 4a^2b^2 \sin^2 C = 4a^2b^2 - 16K^2$

d. $16K^2 = 4a^2b^2 - (a^2 + b^2 - c^2)^2$

e. $16K^2 = [2ab + (a^2 + b^2 - c^2)][2ab - (a^2 + b^2 - c^2)]$

f. $16K^2 = [(a + b)^2 - c^2][c^2 - (a - b)^2]$

g. $16K^2 = [(a + b) + c][(a + b) - c][c + (a - b)][c - (a - b)]$

h. $16K^2 = [2s][2(s - c)][2(s - b)][2(s - a)]$

i. $K = \sqrt{s(s - a)(s - b)(s - c)}$

Mixed Review Exercises

Find the indicated part of $\triangle ABC$ to three significant digits or to the nearest tenth of a degree.

1. $a = 8,\ b = 5,\ \angle C = 62°,\ c = \underline{\ ?\ }$

2. $\angle A = 100°,\ \angle B = 30°,\ b = 15,\ a = \underline{\ ?\ }$

3. $a = 6,\ b = 4,\ c = 9,\ \angle C = \underline{\ ?\ }$

4. $a = 20,\ \angle B = 76°,\ \angle C = 48°,\ b = \underline{\ ?\ }$

Find the five other trigonometric functions of θ.

5. $\sin \theta = \frac{1}{2},\ 90° < \theta < 180°$

6. $\cos \theta = -\frac{3}{4},\ 180° < \theta < 270°$

7. $\tan \theta = -1,\ 270° < \theta < 360°$

8. $\sin \theta = \frac{5}{13},\ 0° < \theta < 90°$

Self-Test 2

Vocabulary solving a triangle (p. 574)
angle of elevation (p. 575)
angle of depression (p. 575)
law of cosines (p. 580)
law of sines (p. 586)
ambiguous case (p. 592)

Give lengths to three significant digits and angle measures to the nearest tenth of a degree.

1. In $\triangle ABC$, $\angle C = 90°$, $\angle B = 27.3°$, and $a = 30$. Find $\angle A$ and sides b and c. **Obj. 12-5, p. 574**

2. In $\triangle DEF$, $d = 18$, $e = 24$ and $\angle F = 42°$. Find side f. **Obj. 12-6, p. 580**

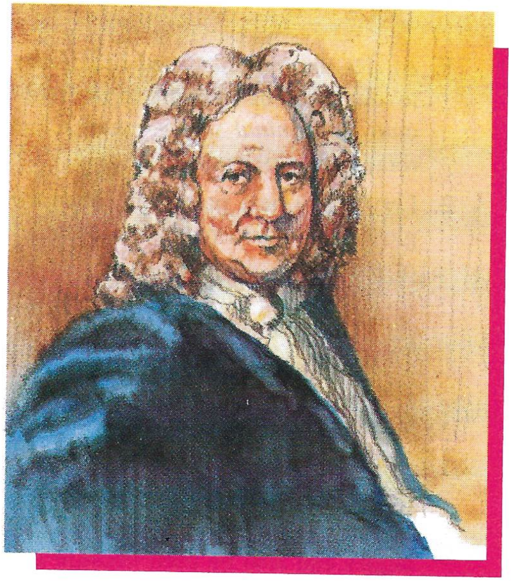
3. In $\triangle XYZ$, $\angle X = 36°$, $x = 14$, and $z = 23.5$. Find $\angle Z$. **Obj. 12-7, p. 586**

4. Solve $\triangle ABC$ if $a = 15$, $b = 12$, and $c = 26$. **Obj. 12-8, p. 591**

5. Find the area of $\triangle RST$ if $r = 9$, $s = 12$, and $\angle T = 53.7°$. **Obj. 12-9, p. 597**

Check your answers with those at the back of the book.

Biographical Note / *Edmund Halley*



A comet usually is visible for only a few weeks or months on its path through the solar system. This fact makes its motion difficult to analyze and predict. Early astronomers debated whether comets moved in straight paths or along some regular curve. The Englishman Edmund Halley (1656–1742) solved this problem by describing comets' orbits as ellipses with predictable paths. He claimed that the comets of 1531, 1607, and 1682 were in fact the same comet and that this comet could be seen from Earth approximately every seventy-five years. Later Halley suggested that the comets reported in 1305, 1380, and 1456 were also sightings of this single comet.

Halley predicted the comet's next return for December, 1758. Off schedule by just a few days, the comet appeared in the same part of the sky he had predicted. Although Halley did not live to see this proof of his theory, the scientific world recognized his achievement by naming the comet after him. Halley's comet was last visible from Earth in 1986 and is not due to be seen again until 2061.

Mathematical problems, as well as the positions of the planets, the size of the universe, and stellar motion, also intrigued Halley. He published papers on topics ranging from higher geometry and the roots of equations to the computation of logarithms and trigonometric functions. One result of his interest in social statistics was the first explanation of how mortality tables could be used to calculate life insurance premiums. Halley was appointed as astronomer royal in 1720. He continued his work well into his old age.

Challenge

One day a visitor to a distant land stood at a fork in the road. The visitor knew that each path led to a different village. The inhabitants of one village always told the truth. The inhabitants of the other village always lied. Luckily, a local citizen happened to be along and agreed to answer one and only one question. The visitor thought for awhile and finally asked a simple question that was guaranteed to indicate the right path to the village of truth tellers, no matter which village the person was from.

What was the question? Explain the visitor's reasoning for asking that particular question.

Chapter Summary

1. Angles may be measured in *degrees,* where one degree is $\frac{1}{360}$ of a complete revolution. The measure is *positive* if the rotation of the angle's *initial side* onto its *terminal side* is counterclockwise. The measure is *negative* if the rotation is clockwise.
2. Let $P(x, y)$ be a point other than the origin on the terminal side of an angle θ in standard position, and let $r = OP$. Then the *trigonometric functions* of θ are:

$$\sin\theta = \frac{y}{r} \qquad \cos\theta = \frac{x}{r} \qquad \tan\theta = \frac{y}{x}$$

$$\csc\theta = \frac{r}{y} \qquad \sec\theta = \frac{r}{x} \qquad \cot\theta = \frac{x}{y}$$

If θ is acute, the numbers x, y, and r can be thought of as being the lengths of the legs and hypotenuse, respectively, of a right triangle.

3. Scientific calculators give good approximations to the values of the trigonometric functions of any angle. Tables giving values of the functions of acute angles can also be used because each angle θ has an acute *reference angle* α such that $\sin\theta = \pm\sin\alpha$, $\cos\theta = \pm\cos\alpha$, and so forth.
4. To *solve a triangle* is to find the measures of all of its sides and angles when three of the measures (including at least one side) are known. Right triangles can be solved using the definitions of the trigonometric functions. *Any* triangle ABC can be solved with the help of the law of cosines (see page 580) and the law of sines (page 586). Depending on the information given, there may be one solution, two solutions, or no solution.
5. The area K of $\triangle ABC$ can be found by using one of the formulas given on page 597.

Chapter Review

Give the letter of the correct answer.

1. Which of the following is coterminal with $-210°$? **12-1**

a. 30° **b.** 150° **c.** $-150°$ **d.** 210°

2. Express 45°14′42″ in decimal degrees.

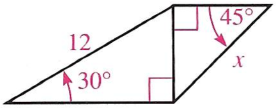
a. 45.452° **b.** 45.542° **c.** 45.245° **d.** 45.45°

3. Find the product of all six trigonometric functions of an angle θ in standard position whose terminal side passes through $(-\sqrt{3}, 1)$. **12-2**

a. -1 **b.** 1 **c.** $\sqrt{3}$ **d.** $\frac{\sqrt{3}}{2}$

4. In $\triangle ABC$, $\angle A = 60°$, $b = 10$, and $\angle B = 90°$. Find c.
a. $5\sqrt{3}$ **b.** 5 **c.** 10 **d.** 20

5. Find x in the diagram at the right.
a. $16\sqrt{2}$ **b.** $6\sqrt{2}$
c. $4\sqrt{3}$ **d.** $8\sqrt{6}$



6. Express cos 230° as a function of an acute angle. 12-3
a. $-\cos 40°$ **b.** $-\sin 40°$ **c.** $-\cos 50°$ **d.** $\cos 50°$

7. Find $\tan \theta$ if $\cos \theta = \frac{8}{17}$ and $\sin \theta > 0$.
a. $\frac{15}{8}$ **b.** $-\frac{15}{17}$ **c.** $-\frac{8}{15}$ **d.** $-\frac{15}{8}$

8. Give the exact value of sin 300°.
a. $-\frac{1}{2}$ **b.** $\frac{1}{2}$ **c.** $-\frac{\sqrt{3}}{2}$ **d.** $\frac{\sqrt{3}}{2}$

9. Find sec 116.7° to four decimal places. 12-4
a. 1.119 **b.** 2.226 **c.** -1.119 **d.** -2.226

10. Find θ to the nearest ten minutes if $\cot \theta = -0.7046$ and $0° < \theta < 180°$.
a. 54°50′ **b.** 125°10′ **c.** 144°50′ **d.** $-56°50'$

11. In $\triangle DEF$, $\angle E = 90°$, $e = 10$, and $d = 8$. Find $\angle F$ to the nearest tenth of a degree. 12-5
a. 36.9° **b.** 53.1° **c.** 38.7° **d.** 44.2°

12. In $\triangle ABC$, $\angle B = 90°$, $\angle C = 55°$, and $c = 12$. Find b to the nearest tenth.
a. 14.6 **b.** 6.9 **c.** 20.9° **d.** 9.8

13. The base angles of an isosceles trapezoid are 45° and the bases have lengths 15 and 25 units. Find the height of the trapezoid.
a. 5 **b.** 5.2 **c.** 10.2 **d.** 5.3

14. In $\triangle ABC$, $a = 6$, $b = 10$, and $c = 7$. Find the measure of the largest angle to the nearest tenth of a degree. 12-6
a. 43.5° **b.** 100.3° **c.** 36.2° **d.** 46.5°

15. Two planes leave the airport at noon, one traveling east at 300 km/h and the other traveling northwest at 450 km/h. How far apart are they at 2 PM?
a. 637 km **b.** 821 km **c.** 1390 km **d.** 1080 km

16. In $\triangle ABC$, find $\angle A$ to the nearest tenth of a degree if $\angle B = 95°$, $b = 15.4$, and $a = 8.0$. 12-7
a. 32.1° **b.** 56.8° **c.** 31.2° **d.** 92.3°

17. Wires of lengths 20 m and 30 m extend from the top of a tower to the ground on the same side of the tower. The shorter wire makes an angle of 42° with the ground. What angle do the wires make with each other?
a. 15.5° **b.** 63.5° **c.** 26.5° **d.** 48.2°

(Chapter Review continues on the next page.)

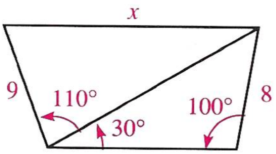
18. In $\triangle ABC$, $\angle B = 30°$ and $c = 20$. For what value(s) of b will the triangle have two solutions? **12-8**

a. $0 < b < 10$ **b.** $10 < b < 20$ **c.** $b < 20$ **d.** $b = 10$

19. Find x in the diagram at the right.

a. 15.8 **b.** 18.4

c. 19.8 **d.** 20.7



20. Find the area of $\triangle ABC$ if $a = 8$, $b = 15$, and $\angle C = 40°$. **12-9**

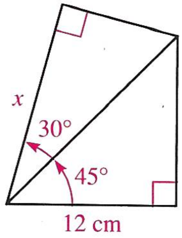
a. 77.1 **b.** 60.0 **c.** 45.9 **d.** 38.6

Chapter Test

1. Express 13.24° in degrees, minutes, and seconds. **12-1**

2. Find two angles, one positive and one negative, that are coterminal with 285°.

3. Find x in the diagram at the right. **12-2**



4. If $\cos \theta = -\frac{3}{4}$ and $\sin \theta < 0$, find $\tan \theta$. **12-3**

5. Give the exact values of the six trigonometric functions of 240°.

6. a. Find sec 145.8°. **12-4**

b. If $\cos \theta = 0.5606$ and $90° < \theta < 360°$, find θ to the nearest tenth of a degree.

7. The angle of elevation from an observer on the street to the top of a building is 55.6°. If the observer is 150 ft from the base of the building, how tall is the building? **12-5**

8. A compass with legs 3 in. long is opened to measure the diameter of a circle. If the diameter is 5 in., what is the angle between the legs of the compass? **12-6**

Give lengths to three significant digits and angle measures to the nearest tenth of a degree.

9. Solve $\triangle DEF$ if $\angle D = 32°$, $\angle E = 108°$, and $f = 12$. **12-7**

10. Solve $\triangle ABC$ if $\angle A = 40°$, $a = 6$, and $b = 8$. **12-8**

11. Find the area of a triangular plot of land if the sides have length 200 m, 150 m, and 100 m. **12-9**

Preparing for College Entrance Exams

Strategy for Success

When solving problems involving trigonometry, area, or distance, you may find it helpful to draw a sketch that shows the given information. Do not make any assumptions in drawing the figure; use only the given information. Use any available space in your test booklet or scrap paper, but avoid making any stray marks on your answer sheet.

Decide which is the best of the choices given and write the corresponding letter on your answer sheet.

1. Find the next term of this sequence: 1, 3, 7, 13, 21, . . .
(A) 27 **(B)** 37 **(C)** 31 **(D)** 33 **(E)** 29

2. Evaluate $64 \sum_{k=1}^{6} 2^{-k}$.
(A) 64 **(B)** 128 **(C)** 63 **(D)** 1 **(E)** $\frac{31}{32}$

3. The infinite series $1 - \frac{5}{4} + \frac{25}{16} - \frac{125}{64} + \ . \ . \ .$
(A) has the common difference $\frac{5}{4}$ **(B)** has the sum $\frac{4}{9}$ **(C)** has no sum
(D) has the sum -4 **(E)** is neither arithmetic nor geometric

4. Which function(s) is (are) not defined for $\phi = -90°$?
I. $\cos \phi$ II. $\sec \phi$ III. $\tan \phi$
(A) II only **(B)** I and II only **(C)** II and III only
(D) I and III only **(E)** I, II, and III

5. If $\csc \phi = 3$ and $90° < \phi < 270°$, find $\cot \phi$.
(A) $-2\sqrt{2}$ **(B)** $\frac{\sqrt{2}}{4}$ **(C)** $-\frac{\sqrt{2}}{4}$ **(D)** -8 **(E)** $-\frac{\sqrt{2}}{3}$

6. In $\triangle XYZ$, $\angle X = 45°$, $\angle Z = 30°$, and $z = 8$. Find x.
(A) $\frac{8}{3}\sqrt{6}$ **(B)** $8\sqrt{2}$ **(C)** $4\sqrt{2}$ **(D)** $4\sqrt{6}$ **(E)** $32\sqrt{2}$

7. Find the area of a triangle with sides of lengths 5, 6, and 7.
(A) $2\sqrt{6}$ **(B)** $3\sqrt{15}$ **(C)** $2\sqrt{14}$ **(D)** $6\sqrt{6}$ **(E)** $\frac{21\sqrt{2}}{2}$

8. In $\triangle RST$, $\angle R = 137.4°$, $t = 15$, and $s = 12$. Find r to the nearest integer.
(A) 10 **(B)** 17 **(C)** 21 **(D)** 23 **(E)** 25