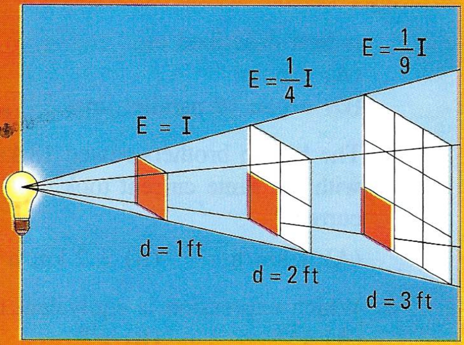
5 Rational Expressions



The illumination (E) on a surface depends directly on the intensity (I) of the light source, but varies according to the square of the distance (d) between the surface and the light source.

Using the Laws of Exponents

5-1 Quotients of Monomials

Objective To simplify quotients using the laws of exponents.

When you multiply fractions, you use the *multiplication rule for fractions*. For example,

$$\frac{3}{5} \cdot \frac{4}{7} = \frac{3 \cdot 4}{5 \cdot 7} = \frac{12}{35}.$$

Multiplication Rule for Fractions

Let p, q, r, and s be real numbers with $q \neq 0$ and $s \neq 0$. Then

$$\frac{p}{q} \cdot \frac{r}{s} = \frac{pr}{qs}.$$

A proof of this rule is asked for in Exercise 33.

Because equality is symmetric, the rule can be rewritten as $\frac{pr}{qs} = \frac{p}{q} \cdot \frac{r}{s}$. If $r = s$, you can replace s by r, obtaining $\frac{pr}{qr} = \frac{p}{q} \cdot \frac{r}{r} = \frac{p}{q} \cdot 1 = \frac{p}{q}$. This proves the following *rule for simplifying fractions*.

Let p, q, and r be real numbers with $q \neq 0$ and $r \neq 0$. Then

$$\frac{pr}{qr} = \frac{p}{q}.$$

Example 1 Simplify: **a.** $\frac{30}{48}$ **b.** $\frac{9xy^3}{15x^2y^2}$

Solution Find the GCF of the numerator and denominator. Then use the rule $\frac{pr}{qr} = \frac{p}{q}$ to simplify the fractions.

a. $\frac{30}{48} = \frac{5 \cdot 6}{8 \cdot \underset{\uparrow \text{GCF}}{6}} = \frac{5}{8}$

b. $\frac{9xy^3}{15x^2y^2} = \frac{3y \cdot 3xy^2}{5x \cdot \underset{\uparrow \text{GCF}}{3xy^2}} = \frac{3y}{5x}$

In Lesson 4-2, you learned the first three laws of exponents listed below. The additional laws, Laws 4 and 5, cover quotients involving exponents.

The Laws of Exponents

Let m and n be positive integers and a and b be real numbers, with $a \neq 0$ and $b \neq 0$ when they are divisors. Then:

1. $a^m \cdot a^n = a^{m+n}$ **2.** $(ab)^m = a^m b^m$ **3.** $(a^m)^n = a^{mn}$

4a. If $m > n$, $\dfrac{a^m}{a^n} = a^{m-n}$ **4b.** If $n > m$, $\dfrac{a^m}{a^n} = \dfrac{1}{a^{n-m}}$

5. $\left(\dfrac{a}{b}\right)^m = \dfrac{a^m}{b^m}$

Here is a concise proof of Law 4(b). We omit many steps that use only the basic properties of real numbers. For example, no reasons are given for the fact that $n = (n - m) + m$.

Statement	*Reason*
1. $a^n = a^{(n-m)+m} = a^{n-m} \cdot a^m$	Law 1 of Exponents
2. $\dfrac{a^m}{a^n} = \dfrac{1 \cdot a^m}{a^{n-m} \cdot a^m}$	Substitution
3. $\dfrac{a^m}{a^n} = \dfrac{1}{a^{n-m}}$	Rule for simplifying fractions

Proofs of Laws 4(a) and 5 are left as Exercises 34 and 35.

Notice how Laws 4(a), 4(b), and 5 are used in Example 2.

Example 2 Simplify: **a.** $\dfrac{2^8}{2^5}$ **b.** $\dfrac{5x^3}{x^7}$ **c.** $\left(\dfrac{t^3}{3}\right)^2$

Solution

a. $\dfrac{2^8}{2^5} = 2^{8-5}$

$= 2^3$

$= 8$ ***Answer***

b. $\dfrac{5x^3}{x^7} = \dfrac{5}{x^{7-3}}$

$= \dfrac{5}{x^4}$ ***Answer***

c. $\left(\dfrac{t^3}{3}\right)^2 = \dfrac{(t^3)^2}{3^2}$

$= \dfrac{t^6}{9}$ ***Answer***

A quotient of monomials having integral coefficients is *simplified* when:

1. the integral coefficients are relatively prime, that is, have no common factor except 1 and -1;
2. each base appears only once; and
3. there are no "powers of powers" [such as $(a^3)^4$].

Example 3 Simplify: **a.** $\dfrac{24s^4t^3}{32st^5}$ **b.** $\dfrac{5x}{4y^2}\left(\dfrac{2y}{x^2}\right)^3$

Solution **a.** *Method 1* Think of each variable separately and use Law 4(a) or 4(b).

$$\frac{24s^4t^3}{32st^5} = \frac{24}{32} \cdot \frac{s^4}{s^1} \cdot \frac{t^3}{t^5} = \frac{3}{4} \cdot \frac{s^3}{1} \cdot \frac{1}{t^2} = \frac{3s^3}{4t^2}$$

Method 2 As in Example 1, find the GCF of the numerator and the denominator. Then simplify the fraction.

$$\frac{24s^4t^3}{32st^5} = \frac{3s^3 \cdot 8st^3}{4t^2 \cdot 8st^3} = \frac{3s^3}{4t^2}$$

b. $\dfrac{5x}{4y^2}\left(\dfrac{2y}{x^2}\right)^3 = \dfrac{5x}{4y^2} \cdot \dfrac{8y^3}{x^6} = \dfrac{5 \cdot 8}{4} \cdot \dfrac{x}{x^6} \cdot \dfrac{y^3}{y^2} = \dfrac{10y}{x^5}$

(This solution used Laws 5 and 3 and then Method 1.)

Oral Exercises

Simplify. Assume that no denominator equals 0 and that variables in exponents represent positive integers.

1. $\dfrac{6^5}{6^3}$
2. $\dfrac{3^5}{3^2}$
3. $\dfrac{(-2)^7}{(-2)^4}$
4. $\dfrac{(-5)^2}{(-5)^4}$
5. $\dfrac{t^8}{t^5}$
6. $\dfrac{x^4}{x^7}$
7. $\dfrac{y^n}{y^{2n}}$
8. $\dfrac{s^{k+2}}{s^k}$
9. $\dfrac{6x^3y^2}{3x^2y}$
10. $\dfrac{4ab^4}{12ab^2}$
11. $\dfrac{(x^2)^3}{x^5}$
12. $\dfrac{(pq)^2}{pq^2}$

Written Exercises

Simplify. Assume that no denominator equals 0.

A

1. $\dfrac{18x^3}{6x}$
2. $\dfrac{5t^3}{15t^5}$
3. $\dfrac{-12p^3q}{4p^2q^2}$
4. $\dfrac{30x^2y^3}{-6x^3y^2}$
5. $\dfrac{-15u^5v^3}{-25u^4v^2}$
6. $\dfrac{48x^5y^5}{32x^4y^6}$
7. $\left(\dfrac{3r}{s^2}\right)^3$
8. $\left(\dfrac{2x^2}{-y}\right)^4$
9. $\dfrac{3s}{t^2} \cdot \dfrac{s^2}{t}$
10. $\dfrac{2u}{v^2} \cdot \dfrac{3u}{2v^2}$
11. $\dfrac{3x^2}{y^2} \cdot \dfrac{3y}{6x}$
12. $\dfrac{xy^2}{2} \cdot \dfrac{6x}{y^2}$
13. $\dfrac{rs^2t^3}{r^3s^2t}$
14. $\dfrac{a^2b^3c}{a^3bc^2}$
15. $\dfrac{u^2}{v}\left(\dfrac{3v}{u^2}\right)^2$
16. $\dfrac{2x^2}{y^3}\left(\dfrac{-y^3}{2x^2}\right)^2$
17. $\dfrac{(4r^2s^2)^2}{(4r^2s)^2}$
18. $\dfrac{(2hk^3)^3}{(-h^2k^2)^2}$
19. $\dfrac{(xyz^2)^2}{(x^2yz)^2}$
20. $\dfrac{(pq^2r^3)^3}{(p^3qr^2)^2}$

B 21. $\left(\frac{2y^2}{3}\right)^2 \cdot \frac{3x}{y^4}$ 22. $\frac{4x^2}{yz^2}\left(\frac{z}{2x}\right)^3$ 23. $\left(\frac{c^3}{d^4}\right)^2\left(\frac{-cd}{h}\right)^3$ 24. $\left(\frac{-4a^2}{3b}\right)^2\left(\frac{-b}{2a}\right)^3$

In Exercises 25–30, assume that no denominator equals 0 and that m and n are integers greater than 1.

25. $\frac{a^{2m}b^{2m+1}}{(a^2b^2)^m}$ 26. $\frac{x^{n+1}y^n}{x^ny^{n-1}}$ 27. $\frac{(pq)^n}{pq^n}$

28. $\frac{(z^n)^3}{z^nz^3}$ 29. $\frac{t^{n+1}t^{n-1}}{t^n}$ 30. $\frac{a^{n-1}b^{2n}}{a^{n+1}(b^2)^{n-1}}$

31. The multiplication rule for fractions can be rewritten in the form $(p \div q) \cdot (r \div s) = (p \cdot r) \div (q \cdot s)$. Replace $\div$ by $-$ and $\cdot$ by $+$. Is the resulting statement true?

32. a. Rewrite the rule $\frac{pr}{qr} = \frac{p}{q}$ using $\cdot$ for multiplication and $\div$ for division.

   b. In your answer to (a) replace $\cdot$ by $+$ and $\div$ by $-$. Is the resulting statement true?

33. Supply reasons for the steps in the following proof of the multiplication rule for fractions.

1.	$\frac{p}{q} \cdot \frac{r}{s} = \left(p \cdot \frac{1}{q}\right)\left(r \cdot \frac{1}{s}\right)$	?
2.	$= pr\left(\frac{1}{q} \cdot \frac{1}{s}\right)$	? and ?
3.	$= pr \cdot \frac{1}{qs}$	?
4.	$= \frac{pr}{qs}$	?

$\therefore \frac{p}{q} \cdot \frac{r}{s} = \frac{pr}{qs}$

34. Supply reasons for the steps in the following proof of Exponent Law 5.

1.	$\left(\frac{a}{b}\right)^m = \underbrace{\frac{a}{b} \cdot \frac{a}{b} \cdot \cdots \cdot \frac{a}{b}}_{m \text{ factors}}$	?
2.	$= \frac{a \cdot a \cdot \cdots \cdot a}{\underbrace{b \cdot b \cdot \cdots \cdot b}_{m \text{ factors}}}$	?

$\therefore \left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$

C 35. Give a concise proof of Exponent Law 4a.

36. In each of the laws of exponents, replace $\cdot$ by $+$, $\div$ by $-$, and, for example, a^m by ma. Is the resulting statement true? (*Example:* $a^m \cdot a^n = a^{m+n}$ becomes $ma + na = (m + n)a$, which is true.)

Mixed Review Exercises

Find the solution set of each inequality.

1. $x^2 + 2x - 8 > 0$
2. $4y + 9 \leq 1$
3. $w^2 < 4$
4. $5 - 3p > -7$
5. $|2t + 5| \leq 3$
6. $2a^2 < a + 3$
7. $-3 < 2c - 1 < 5$
8. $m(m - 1) \geq 2$
9. $|d + 2| > 1$
10. $g \leq g^2$
11. $n^2 + 2 \leq 3n$
12. $-2u < -6$ or $u - 1 \leq 1$

Computer Key-In

If you divide a by b, you obtain a quotient q and a remainder r less than b. For example, $192 \div 84$ gives

$$192 = 84 \cdot 2 + 24$$
$$a = b \cdot q + r.$$

The *Euclidean Algorithm* uses repeated division to find the GCF of two positive integers a and b. To find the GCF of 192 and 84 by the Euclidean Algorithm you would proceed as follows.

$$192 = 84 \cdot 2 + 24$$
$$84 = 24 \cdot 3 + 12$$
$$24 = 12 \cdot 2 + 0$$

The last nonzero remainder, 12, is the GCF of 192 and 84.

Here is a program that uses the Euclidean Algorithm to find GCF's.

```
10 PRINT "FIND THE GCF OF TWO POSITIVE INTEGERS."
20 INPUT "ENTER THE NUMBERS (LARGER FIRST): "; A, B
30 PRINT "THE GCF OF "; A; " AND "; B; " IS ";
40 IF INT (A/B)=A/B THEN 90
50 LET R=A-B*INT (A/B)
60 LET A=B
70 LET B=R
80 GOTO 40
90 PRINT B
95 END
```

Exercises

1. Run the program to find the GCF of each pair of numbers.

a. (432, 528) **b.** (2592, 4682) **c.** (999, 259) **d.** (30,030, 13,122)

2. Run the program to find the GCF of the numerator and the denominator of each fraction. Write each fraction in simplest form.

a. $\frac{87}{1653}$ **b.** $\frac{11,776}{15,872}$ **c.** $\frac{2772}{5940}$ **d.** $\frac{2048}{40,387}$

5-2 Zero and Negative Exponents

Objective To simplify expressions involving the exponent zero and negative integral exponents.

The definition of *power* can be extended so that any integer may be used as an exponent, not just a positive integer. The power a^n when n is 0 or n is a negative integer is defined in such a way that the laws of exponents stated on page 212 continue to hold.

If Law 1, $a^m \cdot a^n = a^{m+n}$, is to hold when $n = 0$, then this statement must be true:

$$a^m \cdot a^0 = a^{m+0} = a^m$$

If $a \neq 0$, you can divide both sides of the resulting equation $a^m \cdot a^0 = a^m$ by a^m to obtain

$$a^0 = \frac{a^m}{a^m} = 1.$$

If Law 1 is to hold for negative exponents, then this statement must be true: If $-n$ is a negative integer and $a \neq 0$,

$$a^{-n} \cdot a^n = a^{-n+n} = a^0 = 1.$$

Since $a^{-n} \cdot a^n = 1$, a^{-n} must be the reciprocal of a^n:

$$a^{-n} = \frac{1}{a^n}$$

The discussion above leads to these definitions.

If n is a positive integer and $a \neq 0$:

$$\mathbf{a^0 = 1} \qquad \mathbf{a^{-n} = \frac{1}{a^n}}$$

The expression 0^0 is not defined.

Example 1 Write in simplest form without negative or zero exponents. Assume that the variables are nonzero.

a. 10^{-2} **b.** $-3c^{-2}$ **c.** $2^{-1}a^0b^{-3}$ **d.** $(3x)^{-2}$

Solution

a. $10^{-2} = \frac{1}{10^2} = \frac{1}{100}$

b. $-3c^{-2} = -3 \cdot \frac{1}{c^2} = -\frac{3}{c^2}$

c. $2^{-1}a^0b^{-3} = \frac{1}{2} \cdot 1 \cdot \frac{1}{b^3}$

$= \frac{1}{2b^3}$

d. $(3x)^{-2} = \frac{1}{(3x)^2}$

$= \frac{1}{9x^2}$

Since $a^{-n} = \frac{1}{a^n}$ and division by zero is not defined, bases of powers with negative exponents must not be zero. To simplify matters we make the following agreement, to hold throughout the rest of this book.

The domains of all variables in any algebraic expression are automatically restricted so that denominators of fractions and bases of powers with negative or zero exponents will not be zero.

For example, in the expression

$$\frac{3a^0b^{-2}}{x^2 - 4},$$

you may assume that $a \neq 0$, $b \neq 0$, $x \neq 2$, and $x \neq -2$ even though these restrictions are not stated.

It can be shown that all the laws of exponents hold even if some of the exponents are negative or zero. For example, Law 3 now implies that

$$(a^3)^{-2} = a^{3(-2)} = a^{-6}.$$

This statement can be justified with the help of Law 3 for *positive* exponents:

$$(a^3)^{-2} = \frac{1}{(a^3)^2} = \frac{1}{a^6} = a^{-6}$$

You can now see that Laws 4(a) and 4(b) give the same results. For example,

by Law 4(a): $\frac{a^2}{a^7} = a^{2-7} = a^{-5} = \frac{1}{a^5}$

by Law 4(b): $\frac{a^2}{a^7} = \frac{1}{a^{7-2}} = \frac{1}{a^5}.$

Example 2 Write in simplest form without negative or zero exponents.

a. $10^{-2} \times 10^{-3}$ **b.** $(-3 \cdot 5^{-1})^{-2}$ **c.** $\frac{3^0x^{-3}y}{2x^{-1}y^{-2}}$

Solution **a.** $10^{-2} \times 10^{-3} = 10^{-2+(-3)} = 10^{-5} = \frac{1}{10^5} = \frac{1}{100,000}$ ***Answer***

b. $(-3 \cdot 5^{-1})^{-2} = (-3)^{-2}(5^{-1})^{-2}$

$= \frac{1}{(-3)^2} \cdot 5^2$

$= \frac{25}{9}$ ***Answer***

c. $\frac{3^0x^{-3}y}{2x^{-1}y^{-2}} = \frac{1 \cdot x^{-3-(-1)}y^{1-(-2)}}{2}$

$= \frac{1x^{-2}y^3}{2}$

$= \frac{y^3}{2x^2}$ ***Answer***

Since a^{-n} and a^n are reciprocals, $\frac{1}{a^{-n}} = a^n$. Example 3 uses this fact.

Example 3 Write in simplest form without negative or zero exponents.

a. $\left(\frac{2}{3}\right)^{-3}$ **b.** $\left(\frac{2x^{-2}}{5y^3}\right)^{-1}$

Solution **a.** $\left(\frac{2}{3}\right)^{-3} = \frac{2^{-3}}{3^{-3}} = 2^{-3} \cdot \frac{1}{3^{-3}} = \frac{1}{2^3} \cdot 3^3 = \frac{3^3}{2^3} = \frac{27}{8}$ ***Answer***

b. $\left(\frac{2x^{-2}}{5y^3}\right)^{-1} = \frac{(2x^{-2})^{-1}}{(5y^3)^{-1}} = \frac{2^{-1}x^2}{5^{-1}y^{-3}} = \frac{5x^2y^3}{2}$ ***Answer***

Negative exponents are introduced in order to write expressions without using fractions. The form of the answer to Example 4(a) below is called *scientific notation*. This notation will be discussed in Lesson 5-3.

Example 4 Write without using fractions: **a.** $\frac{3}{10{,}000}$ **b.** $\frac{5x^2}{yz^3}$

Solution **a.** $\frac{3}{10{,}000} = \frac{3}{10^4} = 3 \times 10^{-4}$ **b.** $\frac{5x^2}{yz^3} = 5x^2y^{-1}z^{-3}$

Oral Exercises

Express in simplest form without negative or zero exponents.

1. $\frac{1}{2^{-3}}$
2. $\frac{5}{3^{-1}}$
3. $10^7 \cdot 10^{-5}$
4. $(3^{-1})^{-2}$
5. $c^{-3} \cdot c^5$
6. $a^{-4} \cdot a^3$
7. $(y^{-1})^{-2}$
8. $y^{-1} \cdot y^{-2}$
9. $(5x)^0$
10. $5x^0$
11. p^2q^{-2}
12. $(p^2q)^{-2}$
13. $\left(\frac{a}{b}\right)^{-2}$
14. $\frac{a}{2^{-1}b^{-2}}$
15. $\frac{x^{-3}}{y^{-3}}$
16. $\left(\frac{r^{-1}}{s^{-1}}\right)^{-1}$

Written Exercises

Write in simplest form without negative or zero exponents.

A

1. $3 \cdot 5^{-1}$
2. $(3 \cdot 5)^{-1}$
3. $(-3^{-1})^{-2}$
4. $(-2^{-2})^{-1}$
5. $(2^{-2} \cdot 3^{-1} \cdot 5^0)^{-1}$
6. $5^{-1}(3^{-2} \cdot 2^{-3})^0$
7. $2\left(\frac{2}{5}\right)^{-2}$
8. $\left(\frac{3}{4}\right)^{-1}\left(\frac{4}{3}\right)^{-2}$

Write without using fractions.

9. $\frac{7}{10{,}000}$ 10. $\frac{3}{1000}$ 11. $\frac{6x^2}{y^3}$ 12. $\frac{x^2}{yz^4}$

Express as a decimal numeral. A calculator may be helpful.

Sample 1 $(-5 \cdot 3^{-2})^{-2} = (-5)^{-2} \cdot 3^4 = \frac{81}{25} = 3.24$

13. 596×10^{-2} 14. 238×10^{-3} 15. 7.2×10^{-3} 16. 1.45×10^{-2}

17. $\left(-\frac{5}{3}\right)^{-3}$ 18. $\left(-\frac{2}{5}\right)^{-2}$ 19. $(3^{-1} \cdot 2^2)^{-2}$ 20. $(-5^{-1} \cdot 2^{-2})^2$

Write in simplest form without negative or zero exponents.

21. $\frac{3x^{-2}}{y^{-1}}$ 22. $\frac{p^{-1}q^{-2}}{p^{-3}}$ 23. $\frac{s^{-2}t^{-3}}{s^{-1}t^0}$

24. $\frac{6xy^{-1}}{-2x^{-2}y^{-1}}$ 25. $\left(\frac{u^{-2}}{v}\right)^{-1}$ 26. $\left(\frac{2}{h^2k^{-3}}\right)^{-2}$

27. $(2x^{-2}y^2)^{-2}$ 28. $\frac{(3x^{-2}y)^{-1}}{(2xy^{-2})^0}$ 29. $3x^2(3xy^{-1})^{-2}$

30. $5t(s^{-1}t^{-2})^{-2}$ 31. $\frac{(2x^{-1})^{-2}}{2(y^{-1})^{-2}}$ 32. $\left(\frac{2pq^{-1}}{4q^2}\right)^{-1}$

33. $\left(\frac{x}{y^2}\right)^{-1}\left(\frac{x^{-2}}{y}\right)^2$ 34. $\left(\frac{3}{t^2}\right)^{-1}\left(\frac{t}{3}\right)^{-2}$ 35. $\left(\frac{p^{-2}q^{-1}}{p^{-1}q^{-2}}\right)^{-1}$

36. $\frac{(ax^2)^{-1}}{a^{-2}x^{-2}}$ 37. $\left(\frac{x^2}{y^{-1}}\right)^{-2}\left(\frac{y^2}{x^{-1}}\right)^2$ 38. $\frac{r^{-2}}{s^2}\left(\frac{1}{rs}\right)^{-2}$

39. $\left(\frac{u}{v^{-1}}\right)^0\left(\frac{u^{-1}}{v^2}\right)^2(uv^2)^{-1}$ 40. $\left(\frac{a^0}{b}\right)^{-2}\left(\frac{a}{b^{-2}}\right)^{-2}$

41. $4x^3y^{-6} + (x^{-1}y^2)^{-3}$ 42. $\left(\frac{u^2}{v}\right)^2 + (-u^{-2}v)^{-2}$

Show by counterexample that the given expressions are *not* equivalent.

Sample 2 $(x + y)^{-2}$ and $x^{-2} + y^{-2}$

Solution Let $x = 1$ and $y = 2$. Then:

$$(x + y)^{-2} = (1 + 2)^{-2} = 3^{-2} = \frac{1}{9} \qquad x^{-2} + y^{-2} = 1^{-2} + 2^{-2} = 1 + \frac{1}{4} = \frac{5}{4} \qquad \left(\frac{1}{9} \neq \frac{5}{4}\right)$$

B 43. $(x + y)^{-1}$ and $x^{-1} + y^{-1}$ 44. $(xy)^{-1}$ and $\frac{x}{y}$

45. xy^{-1} and $\frac{1}{xy}$ 46. $(1 - x)^{-2}$ and $1 - x^{-2}$

In Exercises 47–52, replace the (__?__) by a polynomial to make a true statement.

Sample 3 $x^{-3} + 2x^{-2} - 3x^{-1} = x^{-3}(\underline{\ ?\ })$

Solution $x^{-3} + 2x^{-2} - 3x^{-1} = x^{-3}(1 + 2x - 3x^2)$

47. $x^{-1} - 4x^{-2} + 2x^{-3} = x^{-3}(\underline{\ ?\ })$

48. $2x^{-2} + x^{-1} - 3 = x^{-2}(\underline{\ ?\ })$

49. $4 - 5x^{-1} + x^{-2} = x^{-2}(\underline{\ ?\ })$

50. $x^{-1} - 9x^{-3} = x^{-3}(\underline{\ ?\ })$

C **51.** $x^2(x-1)^{-2} - 4(x-1)^{-1} = (x-1)^{-2}(\underline{\ ?\ })$

52. $(x^2+4)^{-1} - 5(x^2+4)^{-2} = (x^2+4)^{-2}(\underline{\ ?\ })$

In Exercises 53–58, use the exponent laws for positive exponents to prove the forms of the laws given below. In each case, *m* and *n* are positive integers with $m > n$.

53. $a^m a^{-n} = a^{m+(-n)}$

54. $(a^m)^{-n} = a^{m(-n)}$

55. $a^{-m}a^n = a^{(-m)+n}$

56. $\dfrac{a^{-m}}{a^n} = a^{(-m)-n}$

57. $\dfrac{a^m}{a^{-n}} = a^{m-(-n)}$

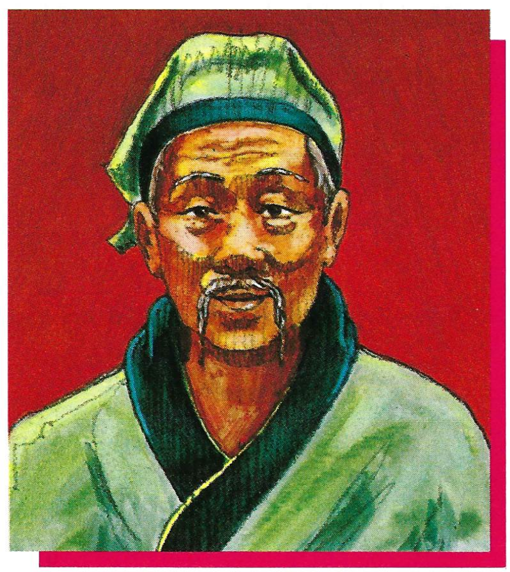
58. $\dfrac{a^{-m}}{a^{-n}} = a^{(-m)-(-n)}$

Biographical Note / *Ch'in Chiu-Shao*

Ch'in Chiu-Shao (ca. 1202–1261) has been called one of the greatest mathematicians of his time. This achievement is particularly remarkable because Ch'in did not devote his life to mathematics. He was accomplished in many other fields and held a series of bureaucratic positions in Chinese provinces.

Ch'in's mathematical reputation rests on one celebrated treatise, *Shu-shu chiu-chang* (''Mathematical Treatise in Nine Sections''), which appeared in 1247. The treatise covers topics ranging from indeterminate analysis to military matters and surveying. Ch'in included a version of the Chinese remainder theorem, which used algorithms to solve problems.

His interest in indeterminate analysis led Ch'in into related fields. He wrote down the earliest explanation of how Chinese calendar experts calculated astronomical data according to the timing of the winter solstice. He also introduced techniques for



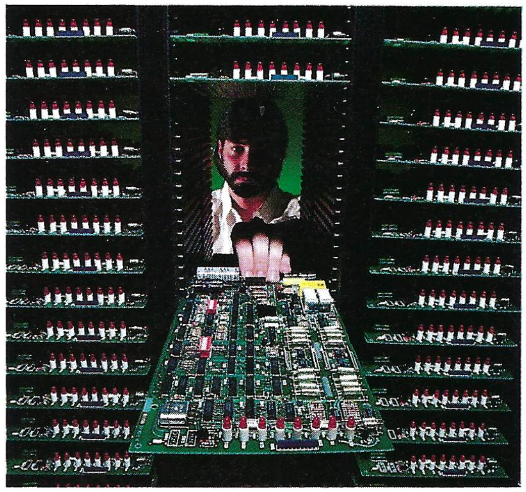
solving equations, finding sums of arithmetic series, and solving linear systems. His use of the zero symbol is a milestone in Chinese mathematics.

5-3 Scientific Notation and Significant Digits

Objective To use scientific notation and significant digits.

In scientific work, you meet very large and very small numbers. For example, the distance that light travels in a year is about 5,680,000,000,000 mi. The time that it takes a signal to travel from one component of a supercomputer to another might be 0.0000000024 s. The wavelength of ultraviolet light is about 0.00000136 cm. To make such numbers easier to work with, you can write them in *scientific notation*:

$$5{,}680{,}000{,}000{,}000 = 5.68 \times 10^{12}$$
$$0.0000000024 = 2.4 \times 10^{-9}$$
$$0.00000136 = 1.36 \times 10^{-6}$$



In **scientific notation,** a number is expressed in the form

$$m \times 10^n$$

where $1 \le m < 10$, and n is an integer.

The digits in the factor m should all be *significant.* A **significant digit** of a number written in decimal form is any nonzero digit or any zero that has a purpose other than placing the decimal point. In the following examples the significant digits are printed in red.

4006   0.4050   320.0   0.00203

In scientific notation

$$4006 = 4.006 \times 10^3, \qquad 0.4050 = 4.050 \times 10^{-1},$$
$$320.0 = 3.200 \times 10^2, \qquad 0.00203 = 2.03 \times 10^{-3}.$$

For a number such as 2300 it is not clear which, if any, of the zeros are significant. Scientific notation eliminates this problem. Writing 2300 as

$$2.3 \times 10^3, \qquad 2.30 \times 10^3, \qquad \text{or } 2.300 \times 10^3$$

indicates, respectively, that none, one, or two of the zeros are significant.

Most measurements are approximate, and the more significant digits that are given in such an approximation, the more *accurate* it is. For example, if a length is given as 2.30×10^3 cm, the measurement has three significant digits. Since the 0 is significant, you know that the length is *not* 2.31×10^3 cm or 2.29×10^3 cm, but it might be 2.302×10^3 cm.

When you multiply or divide measurements, their accuracy limits the accuracy of the answer. *When approximations are multiplied or divided, the product or quotient should have the same number of digits as the least accurate factor.*

Example 1 Write each number in scientific notation. If the number is an integer and ends in zeros, assume that the zeros are not significant.

a. 75,040 **b.** 0.0000702 **c.** 3,050,000

Solution

a. $75{,}040 = 7.504 \times 10^4$ decimal point moved 4 places to the left

b. $0.0000702 = 7.02 \times 10^{-5}$ decimal point moved 5 places to the right

c. $3{,}050{,}000 = 3.05 \times 10^6$ decimal point moved 6 places to the left

Numbers expressed in scientific notation are easy to compare.

Example 2

a. $6.84 \times 10^{-2} > 4.96 \times 10^{-2}$ because $6.84 > 4.96$.

b. $8.7 \times 10^4 < 5.3 \times 10^5$ because $4 < 5$.

c. $5.25 \times 10^{-2} > 7.50 \times 10^{-3}$ because $-2 > -3$.

Example 3 Find a one-significant-digit estimate of x, where

$$x = \frac{5260 \times 0.0682 \times 86.1}{0.420}.$$

Solution

$$x = \frac{5.26 \times 10^3 \times 6.82 \times 10^{-2} \times 8.61 \times 10^1}{4.2 \times 10^{-1}}$$

Write each number in scientific notation.

$$\approx \frac{5 \times 10^3 \times 7 \times 10^{-2} \times 9 \times 10^1}{4 \times 10^{-1}}$$

Round each decimal to a whole number.

$$= \frac{5 \times 7 \times 9}{4} \times 10^{3+(-2)+1-(-1)}$$

Compute, and give the result to one significant digit.

$$= \frac{315}{4} \times 10^3$$

$$\approx 80 \times 10^3 = 8 \times 10^4$$

$$\therefore x \approx 8 \times 10^4, \text{ or } 80{,}000$$

Round off errors may cause the digit obtained in the process shown in Example 3 to be incorrect. (In the example, $x = 73{,}500$ to three significant digits.) Nevertheless, such estimates do describe the "size" of the answer.

Before using a calculator in the exercises, see the Calculator Key-In on page 225 to determine how your calculator handles numbers that are too large or small or too long for the display.

Oral Exercises

Tell how many significant digits each number has.

1. 40.05
2. 10,608
3. 0.050
4. 0.0250
5. 4.0×10^3
6. 7.06×10^{-2}
7. 3.600×10^2
8. 4.8×10^6

Replace the _?_ with < or > to make a true statement.

9. 4.60×10^3 _?_ 3.75×10^3
10. 2.60×10^3 _?_ 6.75×10^4
11. 7.73×10^{-3} _?_ 2.50×10^{-2}
12. 2.78×10^{-4} _?_ 2.78×10^{-6}

Written Exercises

Write each number in scientific notation. If the number is an integer and ends in zeros, assume that the zeros are not significant.

A

1. 7500
2. 106,000
3. 0.608
4. 0.0038
5. 10.05
6. 762.20
7. 0.0320
8. 0.0000460
9. 655×10^3
10. 0.025×10^{-2}
11. 0.560×10^{-3}
12. 4775×10^{-2}

Write each number in decimal form.

13. 5×10^3
14. 10^{-4}
15. 4.3×10^{-3}
16. 10^7
17. 6.75×10^4
18. 6.20×10^{-2}
19. 7.50×10^{-3}
20. 4.000×10^2

Replace the _?_ with < or > to make a true statement.

21. $(2 \times 10^3)^2$ _?_ 5×10^5
22. 1.6×10^{18} _?_ $(1.2 \times 10^9)^2$
23. $(0.005)^3$ _?_ 2×10^{-8}
24. $(2 \times 10^3)^{-2}$ _?_ $(2 \times 10^{-3})^2$

Find a one-significant-digit estimate of each of the following.

B

25. $\dfrac{478 \times 0.230}{0.0281 \times 32.4}$
26. $\dfrac{0.525 \times 7820}{22.5 \times 0.00475}$
27. $\dfrac{73.1 \times (0.493)^2}{0.620 \times (32.6)^2}$
28. $\dfrac{0.000212 \times (588)^2}{57.7 \times (0.0620)^2}$

Simplify, assuming that the factors are approximations. Give answers in scientific notation with the same number of significant digits as in the least accurate factor.

29. $\dfrac{(8 \times 10^6)(2 \times 10^{-2})}{4 \times 10^2}$
30. $\dfrac{(7.5 \times 10^6)(5.0 \times 10^{-1})}{1.5 \times 10^8}$
31. $\dfrac{(8.4 \times 10^{15})(1.5 \times 10^{-5})}{(4.0 \times 10^4)(1.2 \times 10^3)}$

Problems

In these problems, all given data are approximate. In Problems 1–5, give answers reflecting the situation in 1987 using the following table. A calculator may be helpful.

	1987 Population	Land Area (km^2)
U.S.A.	2.44×10^8	9.36×10^6
China	1.06×10^9	9.60×10^6
Italy	5.74×10^7	3.01×10^5
World	5.03×10^9	1.49×10^8

A

1. Write the table above using decimal notation.
2. What percent of the world's land area has **(a)** the United States, **(b)** China, and **(c)** Italy?
3. What percent of the world's population has **(a)** the United States, **(b)** China, and **(c)** Italy?
4. Find the population density (persons per km^2) of **(a)** the United States, **(b)** China, **(c)** Italy, and **(d)** the world.
5. In 1987 the national debt of the United States was about 2.50×10^{12} dollars. How much debt is this per person?
6. How many nanoseconds (1 nanosecond $= 10^{-9}$ s) does it take a computer signal to travel 60 cm at a rate of 2.4×10^{10} cm/s?
7. The designer of a minicomputer wants to limit to 4.0 nanoseconds (see Problem 6) the time that it takes signals to travel from one component to another. What restriction does this put on the distance between components?
8. The estimated masses of a proton and an electron are 1.67×10^{-24} g and 9.11×10^{-28} g, respectively. Find the ratio of the mass of the proton to the mass of the electron.

B

9. The astronomical unit (AU), the light year, and the parsec are units of distance used in astronomy. Copy and complete the following table showing how these units are related.

	AU	parsec	light year
One AU equals	1	?	1.58×10^{-5}
One parsec equals	?	1	?
One light year equals	?	0.307	1

10. Given that 1 light year $= 9.46 \times 10^{12}$ km, express 1 AU and 1 parsec in kilometers. (See Problem 9.)

11. The mean distances from the sun to the planets Mercury and Neptune are 5.79×10^7 km and 4.51×10^9 km, respectively. Express these distances in astronomical units. (See Problems 9 and 10.)

12. A steady current of 1 ampere flowing through a solution of silver nitrate will deposit 1.12×10^{-3} g of silver in one second. How much silver will be deposited by a current of 12.0 amperes in one hour?

13. An electric current of one ampere corresponds to a flow of 6.2×10^{18} electrons per second past any point of the circuit. How many electrons flow through the filament of a 100 watt, 115 volt light bulb during one hour? (*Note:* Watts = amperes × volts)

14. If you were to receive as wages 1 cent for the first day, 2 cents for the second day, 4 cents for the third day, and so on, with each day's wages twice the previous day's, about how many dollars would you receive on the 31st day? Use the approximation $2^{10} \approx 10^3$ and express your answer in decimal form.

C 15. The masses of the sun and Earth are 2.0×10^{30} kg and 6.0×10^{24} kg, respectively, and their radii are 7.0×10^8 m and 6.4×10^6 m. Show that the average density of Earth is about four times that of the sun. (*Note:* Density = mass ÷ volume)

Mixed Review Exercises

Simplify.

1. $\dfrac{18x^3y}{27x^2y^4}$
2. $\left(\dfrac{3a}{2b}\right)^2$
3. $\dfrac{4m}{n} \cdot \dfrac{m^2}{5n}$
4. $\dfrac{(c^2d)^3}{(cd^2)^3}$
5. $\dfrac{2p^2}{q} \cdot \dfrac{q^3}{2p^3}$
6. $\dfrac{(-w)^2}{-w^2}$
7. $\dfrac{14z^5}{10z^5}$
8. $\dfrac{u^4}{v}\left(\dfrac{v^2}{u^3}\right)^2$

Express in simplest form without negative or zero exponents.

9. $(x^{-3}y^2)^{-1}$
10. $\left(\dfrac{c^{-1}}{d}\right)^{-2}$
11. $(6m^2)(2m)^{-2}$
12. $\dfrac{a^{-4}b}{a^2b^{-3}}$

Calculator Key-In

Some calculators will indicate an error when the result of a computation has too many digits to be shown in full on the display. Others will display such a number in scientific notation, with the exponent of 10 shown at the right. Most scientific calculators have a key that allows the user to select scientific notation for all numbers displayed and to enter numbers in scientific notation.

When the result of a computation has more significant digits than the display can show, the number may be *truncated* (the extra digits are discarded) or *rounded* (the last place shown is rounded to the correct value).

You can find answers to some of the exercises below without a calculator, but doing them on your calculator will help you find out how your calculator handles numbers that are too large or small or too long for the display.

Evaluate on your calculator.

1. $20{,}000 \times 50{,}000$

2. $352{,}129 \times 416{,}187$

3. $(0.00001)^2$

Find the value of each expression to four significant digits. Give your answers in scientific notation.

4. $(4.325 \times 10^5)(9.817 \times 10^{12})$

5. $(1.589 \times 10^3)(6.805 \times 10^{-8})$

6. $2.563 \div 493{,}412$

7. $\dfrac{(4.303 \times 10^6)(2.115 \times 10^3)}{9.563 \times 10^8}$

8. a. Find $\frac{2}{3}$ as a decimal on your calculator.

b. From part (a), tell whether your calculator truncates or rounds.

Self-Test 1

Vocabulary scientific notation (p. 221) significant digit (p. 221)

Simplify.

1. $\dfrac{24s^3t^2}{16st^3}$

2. $\left(\dfrac{2x^2y}{w^2}\right)^2\left(\dfrac{y^2w}{x^2}\right)^3$

Obj. 5-1, p. 211

Write in simplest form without negative or zero exponents.

3. $\dfrac{p^{-2}qr^{-1}}{p^0q^{-2}r^{-3}}$

4. $\left(\dfrac{x^2y^{-1}}{z}\right)^{-2}\left(\dfrac{x^2y^2}{z^{-3}}\right)$

Obj. 5-2, p. 216

Replace the $\underline{\ ?\ }$ by a polynomial to make a true statement.

5. $x^{-1} - 4x^{-3} = x^{-3}\,(\underline{\ ?\ })$

Write each number in scientific notation.

6. 482.60

7. 0.000210

Obj. 5-3, p. 221

Write each number in decimal form.

8. 3.609×10^3

9. 5.400×10^{-2}

10. Give a one-significant-digit estimate of $\dfrac{683 \times 0.536}{0.00392}$.

Rational Expressions

5-4 Rational Algebraic Expressions

Objective To simplify rational algebraic expressions.

You know that a *rational number* is one that can be expressed as a quotient of integers. Similarly, a **rational algebraic expression,** or **rational expression,** is one that can be expressed as a quotient of polynomials. Some examples of rational expressions are

$$\frac{4xy^2}{7y}, \quad \frac{x^2 - 3x - 4}{x^2 - 1}, \quad \text{and} \quad x(x^2 - 4)^{-1} = \frac{x}{x^2 - 4}.$$

A rational expression is *simplified,* or in *simplest form,* when it is expressed as a quotient of polynomials whose greatest common factor is 1. (Recall Lesson 4-6.) To simplify a rational expression you factor the numerator and denominator and then look for common factors.

Example 1 Simplify $\frac{x^2 - 2x}{x^2 - 4}$.

Solution

$$\frac{x^2 - 2x}{x^2 - 4} = \frac{x\cancel{(x - 2)}}{(x + 2)\cancel{(x - 2)}} \qquad \text{Factor the numerator and the denominator.}$$

$$= \frac{x}{x + 2} \qquad \text{Simplify.}$$

Answer

Factors of the numerator and denominator may be opposites of each other, as in Example 2.

Example 2 Simplify $(3x - 5x^2 - 2x^3)(6x^2 - 5x + 1)^{-1}$.

Solution

$$(3x - 5x^2 - 2x^3)(6x^2 - 5x + 1)^{-1} = \frac{3x - 5x^2 - 2x^3}{6x^2 - 5x + 1} \qquad \text{Write as a fraction.}$$

$$= \frac{x(3 + x)(1 - 2x)}{(2x - 1)(3x - 1)} \qquad \text{Factor.}$$

$$= \frac{x(3 + x)(-1)\cancel{(2x - 1)}}{\cancel{(2x - 1)}(3x - 1)} \qquad \text{Replace } 1 - 2x \text{ by } (-1)(2x - 1).$$

$$= \frac{-x(3 + x)}{3x - 1}, \text{ or } -\frac{x(x + 3)}{3x - 1}$$

Answer

A function that is defined by a simplified rational expression in one variable is called a **rational function.**

Example 3 Let $f(x) = \dfrac{2x^2 - 7x + 3}{x^3 + x^2 - 2x}$.

a. Find the domain of f.

b. Find the zeros of f, if any.

Solution

$$f(x) = \frac{2x^2 - 7x + 3}{x^3 + x^2 - 2x} = \frac{(2x - 1)(x - 3)}{x(x - 1)(x + 2)}$$

The zero-product property is used for both (a) and (b).

a. According to the agreement made on page 217, the domain of f consists of all real numbers except 0, 1, and -2.

b. $f(x) = 0$ if and only if $(2x - 1)(x - 3) = 0$. Therefore, the zeros of f are $\frac{1}{2}$ and 3.

Oral Exercises

Simplify.

1. $\dfrac{5x^2y^2}{xy^3}$ **2.** $\dfrac{5x^2y^3}{3xy^2}$ **3.** $\dfrac{x(x - 2)}{x(x + 2)}$

4. $\dfrac{(x - 1)(x + 2)}{2 + x}$ **5.** $\dfrac{x^2 - x}{x - 1}$ **6.** $\dfrac{2 - x}{x - 2}$

Find (a) the domain of each rational function f and (b) the zeros of f, if any.

7. $f(x) = \dfrac{x - 2}{x}$ **8.** $f(x) = \dfrac{x}{x - 2}$ **9.** $f(x) = \dfrac{x + 1}{x - 1}$

10. $f(x) = \dfrac{x^2 + 1}{x + 1}$ **11.** $f(x) = \dfrac{x + 3}{x^2 + 9}$ **12.** $f(x) = \dfrac{(x - 1)(x - 3)}{x - 2}$

13. $f(x) = \dfrac{x^2 + 1}{x^2 - 1}$ **14.** $f(x) = \dfrac{x^2 - 4}{x^2 + 4}$ **15.** $f(x) = \dfrac{x - 4}{x^2 - 4}$

Written Exercises

Simplify.

A **1.** $\dfrac{5x^2 - 15x}{10x^2}$ **2.** $\dfrac{3t^4 - 9t^3}{6t^2}$

3. $\dfrac{u^2 - u - 2}{u^2 + u}$ **4.** $\dfrac{z^3 - 4z}{z^2 - 4z + 4}$

5. $(p - q)(q - p)^{-1}$

6. $(r^2 - rs)(r^2 - s^2)^{-1}$

7. $\dfrac{s^2 - t^2}{(t - s)^2}$

8. $\dfrac{(a - x)^2}{x^2 - a^2}$

9. $\dfrac{x^2 - 5x + 6}{x^2 - 7x + 12}$

10. $\dfrac{2t^2 + 5t - 3}{2t^2 + 7t + 3}$

11. $\dfrac{6y^2 - 5y + 1}{1 - y - 6y^2}$

12. $\dfrac{9 - 4z^2}{6z^2 - 5z - 6}$

13. $(r^2 - 5r + 4)(r - 4)^{-2}$

14. $(p^2 + 4p - 5)(p - 1)^{-2}$

15. $\dfrac{x^2 + 2x - 8}{(2 - x)(4 + x)}$

16. $\dfrac{(t + 1)(t^2 - 1)}{(t - 1)(t + 1)^2}$

17. $(z^4 - 1)(z^4 - z^2)^{-1}$

18. $(1 - r^3)(1 - r)^{-3}$

19. $\dfrac{(x^2 - a^2)^2}{(x - a)^2}$

20. $\dfrac{t^4 - c^4}{(t + c)^2(t^2 + c^2)}$

Find (a) the domain of each function and (b) its zeros, if any.

21. $f(t) = \dfrac{t^2 - 9}{t^2 - 9t}$

22. $g(x) = \dfrac{x^3 + 2x}{x^2 - 4}$

23. $F(x) = (x^4 - 16)(x^3 - 1)^{-1}$

24. $h(y) = (y^3 - 8)(y + 2)^{-3}$

25. $g(t) = \dfrac{2t^2 + 3t - 9}{t^3 - 4t}$

26. $G(s) = \dfrac{4s^2 + 15s - 4}{(2s - 1)^2}$

B 27. $f(x) = \dfrac{x^3 - 2x^2 + x - 2}{x^4 + x^2 - 2}$

28. $h(t) = \dfrac{t^3 + 4t^2 - t - 4}{t^3 - t^2 + t - 1}$

Simplify.

29. $\dfrac{x^3 + x^2 - x - 1}{x^3 - x^2 - x + 1}$

30. $\dfrac{t^4 - 1}{t^3 + t^2 + t + 1}$

31. $\dfrac{x^3 - x^2y + xy^2 - y^3}{x^4 - y^4}$

32. $\dfrac{x^4 - 2x^2y^2 + y^4}{x^4 - x^3y - xy^3 + y^4}$

33. $\dfrac{s^4 - t^4}{s^4 - 2s^2t^2 + t^4}$

34. $\dfrac{u^4 - v^4}{u^4 + 2u^2v^2 + v^4}$

35. $\dfrac{x^4 + x^3y - xy^3 - y^4}{x^4 - y^4}$

36. $\dfrac{x^2 - y^2 - 4x + 4y}{x^2 - y^2 + 4x - 4y}$

37. $\dfrac{x^2 - y^2 - 4y - 4}{x^2 - y^2 - 4x + 4}$

38. $\dfrac{ax + by - bx - ay}{ax - by + bx - ay}$

C 39. $\dfrac{x^2 - y^2 - z^2 - 2yz}{x^2 - y^2 + z^2 - 2xz}$

40. $\dfrac{x^2 + y^2 - z^2 - 2xy}{x^2 - y^2 + z^2 - 2xz}$

41. $\dfrac{x^4 + x^2y^2 + y^4}{x^3 + y^3}$ (*Hint:* See Exercise 57, p. 186.)

42. $\dfrac{x^{2n} + 2x^ny^n - 3y^{2n}}{x^{2n} + 5x^ny^n + 6y^{2n}}$

43. $\dfrac{x^{2n} - 2x^ny^n + y^{2n}}{x^{2n} + 3x^ny^n - 4y^{2n}}$

Graphing Rational Functions

To graph a rational function, $f(x) = \dfrac{x}{x-2}$ for example, you need to plot points (x, y) for which $y = \dfrac{x}{x-2}$. As you will see, the graph of a rational function can approach, without intersecting, certain lines called *asymptotes*.

Example 1 Graph $f(x) = \dfrac{x}{x-2}$.

Solution The domain of f consists of all real numbers except 2. Using a calculator, prepare two tables of x-y values: one for $x < 2$ and one for $x > 2$.

$x < 2$

x	$y = \dfrac{x}{x-2}$
-2	$\dfrac{-2}{-2-2} = 0.5$
-1	$\dfrac{-1}{-1-2} \approx 0.33$
0	$\dfrac{0}{0-2} = 0$
1	$\dfrac{1}{1-2} = -1$
1.5	$\dfrac{1.5}{1.5-2} = -3$
1.9	$\dfrac{1.9}{1.9-2} = -19$
1.99	$\dfrac{1.99}{1.99-2} = -199$

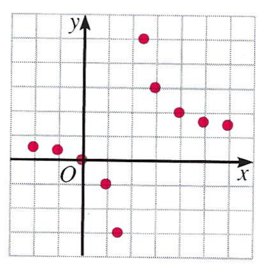
As x approaches 2 from the left, y becomes increasingly small.

$x > 2$

x	$y = \dfrac{x}{x-2}$
6	$\dfrac{6}{6-2} = 1.5$
5	$\dfrac{5}{5-2} \approx 1.67$
4	$\dfrac{4}{4-2} = 2$
3	$\dfrac{3}{3-2} = 3$
2.5	$\dfrac{2.5}{2.5-2} = 5$
2.1	$\dfrac{2.1}{2.1-2} = 21$
2.01	$\dfrac{2.01}{2.01-2} = 201$

As x approaches 2 from the right, y becomes increasingly large.

Some of the points from the two tables are plotted on the grid at the right. Notice that for values of x near 2, the points on the graph of f "follow" the vertical line $x = 2$. The line $x = 2$ is an asymptote of the graph of f.

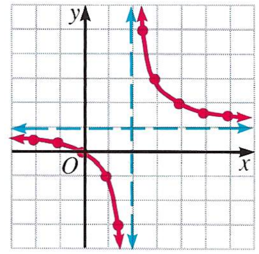


The graph of f has another asymptote. As $|x|$ gets increasingly large, $f(x)$ approaches 1:

$$f(12) = 1.2, \qquad f(102) = 1.02, \qquad f(1002) = 1.002;$$
$$f(-8) = 0.8, \qquad f(-98) = 0.98, \qquad f(-998) = 0.998.$$

Therefore the horizontal line $y = 1$ is also an asymptote of the graph of f.

To obtain the complete graph of f, draw the asymptotes $x = 2$ and $y = 1$ as dashed lines and connect the points already plotted with a smooth curve, as shown at the right.



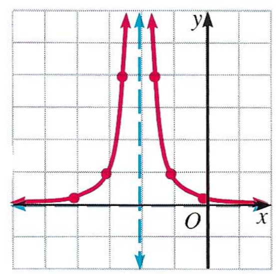
Example 2 Graph $f(x) = \frac{1}{(x+2)^2}$.

Solution The domain of f consists of all real numbers except -2. Because $f(x)$ gets increasingly large as x approaches -2, the line $x = -2$ is a vertical asymptote of the graph of f.

As $|x|$ gets increasingly large, $f(x)$ approaches 0. Therefore the x-axis is a horizontal asymptote of the graph of f.

A table of x-y values and the graph of f are shown below.

x	$y = \frac{1}{(x+2)^2}$
-4	0.25
-3	1
-2.5	4
-1.5	4
-1	1
0	0.25



Exercises

Graph each function. Show any asymptotes as dashed lines. You may wish to check your graphs on a computer or a graphing calculator.

A

1. $f(x) = \frac{1}{x}$ **2.** $g(x) = -\frac{1}{x}$ **3.** $h(x) = \frac{x+1}{x-1}$

4. $f(x) = \frac{x-1}{x+1}$ **5.** $g(x) = \frac{2x}{x-3}$ **6.** $h(x) = \frac{3x}{x+2}$

B

7. $f(x) = \frac{1}{x^2}$ **8.** $g(x) = \frac{1}{(x-1)^2}$ **9.** $h(x) = \frac{x^2+1}{x^2}$

10. $f(x) = \frac{1-x^2}{x^2}$ **11.** $g(x) = \frac{10}{x^2+1}$ **12.** $h(x) = \frac{10x^2}{x^2+1}$

5-5 Products and Quotients of Rational Expressions

Objective To multiply and divide rational expressions.

To find the product of two or more rational expressions, you use the multiplication rule for fractions: $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$. Products of rational expressions should always be expressed in simplest form.

Example 1 Simplify $\frac{3x^2 - 6x}{x^2 - 6x + 9} \cdot \frac{x^2 - x - 6}{x^2 - 4}$.

Solution
$$\frac{3x^2 - 6x}{x^2 - 6x + 9} \cdot \frac{x^2 - x - 6}{x^2 - 4} = \frac{3x(x-2)}{(x-3)(x-3)} \cdot \frac{\cancel{(x+2)}(x-3)}{\cancel{(x+2)}(x-2)}$$
$$= \frac{3x\cancel{(x-2)}\cancel{(x-3)}}{(x-3)\cancel{(x-3)}\cancel{(x-2)}}$$
$$= \frac{3x}{x-3} \quad \textit{Answer}$$

By the definition of division (page 33), a quotient is the product of the dividend and the reciprocal of the divisor. The reciprocal of $\frac{r}{s}$ is $\frac{s}{r}$ because $\frac{r}{s} \cdot \frac{s}{r} = 1$. Combining these facts gives the following rule.

Division Rule for Fractions

Let p, q, r, and s be real numbers with $q \neq 0$, $r \neq 0$, and $s \neq 0$. Then
$$\frac{p}{q} \div \frac{r}{s} = \frac{p}{q} \cdot \frac{s}{r}.$$

Example 2 Simplify.

a. $\frac{14}{15} \div \frac{7}{5}$ **b.** $\frac{6xy}{a^2} \div \frac{3y}{a^3x}$

Solution

a. $\frac{14}{15} \div \frac{7}{5} = \frac{14}{15} \cdot \frac{5}{7} = \frac{14 \cdot 5}{15 \cdot 7} = \frac{2}{3}$ ***Answer***

b. $\frac{6xy}{a^2} \div \frac{3y}{a^3x} = \frac{6xy}{a^2} \cdot \frac{a^3x}{3y} = \frac{6a^3x^2y}{3a^2y} = 2ax^2$ ***Answer***

When multiplying rational expressions, you can divide out factors common to a numerator and a denominator *before* you write the product as a single fraction. For instance, for the product in Example 2(b),

$$\frac{\overset{2}{\cancel{6}}x\cancel{y}}{\underset{1}{\cancel{a^2}}} \cdot \frac{\overset{a}{\cancel{a^3}}x}{\underset{1\,1}{\cancel{3}\cancel{y}}} = 2ax^2.$$

Sometimes you may find it helpful to rewrite a quotient using the ÷ sign.

Example 3 Simplify $\dfrac{\dfrac{a^2 - 4ab + 3b^2}{a + 2b}}{a^2 - ab - 6b^2}$.

Solution

$$\begin{aligned}\frac{\dfrac{a^2 - 4ab + 3b^2}{a + 2b}}{a^2 - ab - 6b^2} &= \frac{a^2 - 4ab + 3b^2}{a + 2b} \div \frac{a^2 - ab - 6b^2}{1}\\ &= \frac{a^2 - 4ab + 3b^2}{a + 2b} \cdot \frac{1}{a^2 - ab - 6b^2}\\ &= \frac{(a - b)\cancel{(a - 3b)}}{a + 2b} \cdot \frac{1}{(a + 2b)\cancel{(a - 3b)}}\\ &= \frac{a - b}{(a + 2b)^2} \quad \textbf{\textit{Answer}}\end{aligned}$$

When you simplify expressions, perform multiplication and division in order from left to right, as in Example 4.

Example 4 Simplify $\dfrac{6p^2q}{r} \div \dfrac{3pq^2}{r} \cdot \dfrac{2q^2}{pr}$.

Solution

$$\begin{aligned}\frac{6p^2q}{r} \div \frac{3pq^2}{r} \cdot \frac{2q^2}{pr} &= \left(\frac{6p^2q}{r} \div \frac{3pq^2}{r}\right) \cdot \frac{2q^2}{pr}\\ &= \frac{6p^2q}{r} \cdot \frac{r}{3pq^2} \cdot \frac{2q^2}{pr}\\ &= \frac{4q}{r} \quad \textbf{\textit{Answer}}\end{aligned}$$

Oral Exercises

Simplify.

1. $\frac{4}{5} \cdot 2$ **2.** $\frac{4}{5} \div 2$ **3.** $\frac{4}{5} \div \frac{1}{2}$ **4.** $\frac{4}{5} \cdot \frac{1}{2}$

5. $\frac{a^2}{b} \div \frac{a}{b^2}$ **6.** $\frac{a^2}{b} \cdot \frac{a}{b^2}$ **7.** $\frac{3s}{4t} \cdot \frac{2t}{s^2}$ **8.** $\frac{3s}{4t} \div \frac{2t}{s^2}$

Written Exercises

Simplify. Write answers without negative or zero exponents.

A

1. $\frac{-10}{21} \div \frac{15}{28}$
2. $-\frac{26}{25} \div \frac{39}{20}$
3. $\frac{5x^3}{-3} \cdot \frac{-6}{10x^2}$
4. $\frac{22z^2}{15} \cdot \frac{-5}{11z^3}$
5. $\frac{8t^2}{3} \div \frac{2t}{9}$
6. $\frac{28x}{25} \div \frac{21x^3}{15}$
7. $\frac{x^2}{4} \cdot \left(\frac{xy}{6}\right)^{-1} \cdot \frac{2y^2}{x}$
8. $2uv \div \frac{2u^2}{v} \div \frac{2v^2}{u}$
9. $\frac{4rs^2}{45} \div \frac{8s}{27r} \div \frac{9rs}{10}$
10. $\frac{7x^2}{9y} \div \frac{4x}{15y^2} \cdot \left(\frac{35}{6xy}\right)^{-1}$
11. $\frac{x(x-1)}{(x-2)^2} \div \frac{(x-1)^2}{x-2}$
12. $\frac{t-2}{t+3} \cdot \frac{t^2+2t-3}{t^2-t-2}$
13. $\frac{4u^2-1}{u^2-4} \cdot \frac{u-2}{2u-1}$
14. $\frac{x^2}{x-1} \cdot \frac{x+1}{x+2} \div \frac{x}{(x-1)(x+2)}$
15. $\frac{x^2-4}{2x^2-5x+2} \div \frac{2x^2-3x-2}{4x^2-1}$
16. $\frac{3x^2-8x+4}{9x^2-4} \div \frac{3x^2-5x-2}{9x^2-3x-2}$

B

17. $\dfrac{\frac{p^4-q^4}{(p+q)^2}}{p^2+q^2}$
18. $\dfrac{\frac{x^2-y^2}{x+y}}{x^4-y^4}$
19. $\frac{u^2v}{u+v} \div (u+v) \cdot \frac{u^2+2uv+v^2}{uv^2-u^2v}$
20. $\frac{x^2+3ax}{3a-x} \cdot \frac{x^2-4ax+3a^2}{a^2-x^2} \div \frac{x+3a}{x+a}$
21. $\frac{3x^2+xy-2y^2}{3x^2-xy-2y^2} \div \frac{3x^2+7xy-6y^2}{3x^2-2xy-y^2} \div \frac{3x+y}{3x+2y}$
22. $\frac{r^2+4rs+3s^2}{r^2+5rs+6s^2} \cdot (r+2s)^{-1} \div \frac{r+s}{r^2+4rs+4s^2}$
23. $(a^4+2a^2b^2+b^4) \div (a^4-b^4) \cdot (a-b)$

C

24. $(u^4+u^2v^2+v^4) \div (u^6-v^6) \cdot (u^2-v^2)$

Mixed Review Exercises

Express in scientific notation.

1. 0.00054
2. 63,400,000
3. 0.1
4. 3281

Simplify.

5. $\frac{x^2+x-6}{x^2-x-2}$
6. $\frac{15y^2}{15y^3-10y^2}$
7. $\frac{16c^3d^2}{24cd^5}$
8. $\frac{a^2-1}{a^2+2a+1}$
9. $\frac{3m+12}{2m+8}$
10. $\frac{(6z^2)^2}{(4z^3)^3}$
11. $\frac{4t^2+4}{8t^2-8}$
12. $(u^3+1)(u+1)^{-1}$

5-6 Sums and Differences of Rational Expressions

Objective To add and subtract rational expressions.

The definition of division and the distributive property can be used to prove the following rules for adding and subtracting fractions with *equal* denominators.

$$\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c} \qquad \frac{a}{c} - \frac{b}{c} = \frac{a-b}{c}$$

These rules can be extended to more than two terms.

Example 1 Simplify: **a.** $\frac{4}{15} + \frac{13}{15} - \frac{7}{15}$ **b.** $\frac{3x^2 - 8}{x + 2} - \frac{x^2 - 9}{x + 2}$

Solution **a.** $\frac{4}{15} + \frac{13}{15} - \frac{7}{15} = \frac{4 + 13 - 7}{15} = \frac{10}{15} = \frac{2}{3}$ ***Answer***

b. $\frac{3x^2 - 8}{x + 2} - \frac{x^2 - 9}{x + 2} = \frac{3x^2 - 8 - x^2 + 9}{x + 2}$

$= \frac{2x^2 + 1}{x + 2}$ ***Answer***

To add or subtract fractions with *different* denominators, use this rule.

Rule for Adding and Subtracting Fractions

1. Find the **least common denominator (LCD)** of the fractions (that is, the least common multiple (LCM) of their denominators).
2. Express each fraction as an equivalent fraction with the LCD as denominator.
3. Add or subtract and then simplify the result.

Example 2 Simplify $\frac{25}{42} + \frac{11}{18} - 2$.

Solution The LCM of $42 = 2 \cdot 3 \cdot 7$ and $18 = 2 \cdot 3^2$ is $2 \cdot 3^2 \cdot 7 = 126$. So the LCD is 126.

$$\frac{25}{42} + \frac{11}{18} - 2 = \frac{25 \cdot 3}{42 \cdot 3} + \frac{11 \cdot 7}{18 \cdot 7} - \frac{2 \cdot 126}{1 \cdot 126}$$

$$= \frac{75}{126} + \frac{77}{126} - \frac{252}{126}$$

$$= -\frac{100}{126} = -\frac{50}{63}$$ ***Answer***

Recall (page 190) that the least common multiple (LCM) of two or more polynomials is the common multiple having the least degree and least positive common factor.

Example 3 Simplify $\frac{1}{6a^2} - \frac{1}{2ab} + \frac{3}{8b^2}$.

Solution The LCM of $6a^2$, $2ab$, and $8b^2$ is $24a^2b^2$. So the LCD is $24a^2b^2$.

$$\begin{aligned}\frac{1}{6a^2} - \frac{1}{2ab} + \frac{3}{8b^2} &= \frac{1 \cdot 4b^2}{6a^2 \cdot 4b^2} - \frac{1 \cdot 12ab}{2ab \cdot 12ab} + \frac{3 \cdot 3a^2}{8b^2 \cdot 3a^2}\\ &= \frac{4b^2}{24a^2b^2} - \frac{12ab}{24a^2b^2} + \frac{9a^2}{24a^2b^2}\\ &= \frac{4b^2 - 12ab + 9a^2}{24a^2b^2}\\ &= \frac{(2b - 3a)^2}{24a^2b^2} \quad \textit{Answer}\end{aligned}$$

When you add fractions in which the denominators are trinomials, factoring these trinomials will often shorten the work.

Example 4 Simplify $\frac{3}{x^2 + x - 6} - \frac{2}{x^2 - 3x + 2}$.

Solution $x^2 + x - 6 = (x - 2)(x + 3)$; $x^2 - 3x + 2 = (x - 1)(x - 2)$
So the LCD is $(x - 1)(x - 2)(x + 3)$.

$$\begin{aligned}\frac{3}{x^2 + x - 6} - \frac{2}{x^2 - 3x + 2} &= \frac{3}{(x - 2)(x + 3)} - \frac{2}{(x - 1)(x - 2)}\\ &= \frac{3(x - 1)}{(x - 1)(x - 2)(x + 3)} - \frac{2(x + 3)}{(x - 1)(x - 2)(x + 3)}\\ &= \frac{3x - 3 - 2x - 6}{(x - 1)(x - 2)(x + 3)}\\ &= \frac{x - 9}{(x - 1)(x - 2)(x + 3)} \quad \textit{Answer}\end{aligned}$$

Oral Exercises

Simplify.

1. $\frac{3}{5} + \frac{1}{5}$ **2.** $\frac{5}{7} - \frac{2}{7}$ **3.** $\frac{5}{6} - \frac{1}{6}$ **4.** $\frac{3}{8} + \frac{1}{4}$

5. $\frac{1}{3} + \frac{1}{3a}$ **6.** $\frac{2}{x} - \frac{1}{2x}$ **7.** $\frac{1}{x} + \frac{1}{y}$ **8.** $z + \frac{1}{z}$

Written Exercises

Simplify.

A

1. $\frac{5}{16} - \frac{12}{16} + \frac{3}{16}$
2. $\frac{7}{10} + \frac{11}{20} - \frac{9}{20}$
3. $\frac{1}{2} - \frac{3}{7} + \frac{5}{14}$
4. $\frac{2}{3} - \frac{3}{5} + \frac{4}{15}$
5. $\frac{5}{12} + \frac{5}{18} - \frac{5}{36}$
6. $\frac{7}{12} - 1 + \frac{19}{20}$
7. $\frac{t+2}{3} + \frac{t-4}{6}$
8. $\frac{x+3}{5} - \frac{2x+1}{10}$
9. $\frac{z-1}{z} + \frac{z+1}{z^2}$
10. $\frac{x+2}{x^2} + \frac{x-2}{2x}$
11. $\frac{t-4}{2t} - \frac{t-6}{3t}$
12. $\frac{2x+5}{4x^2} + \frac{2x-5}{10x}$
13. $\frac{1}{2pq^4} + \frac{2}{p^3q^2}$
14. $\frac{1}{xy^3} - \frac{1}{x^3y}$
15. $ab^{-1} - a^{-1}b$
16. $4s^{-2} - (4t)^{-2}$
17. $\frac{a}{bc} + \frac{b}{ac} + \frac{c}{ab}$
18. $\frac{1}{yz} + \frac{1}{zx} + \frac{1}{xy}$
19. $\frac{y-z}{yz} - \frac{z-x}{zx} - \frac{x-y}{xy}$
20. $\frac{1}{x^2} + \frac{1}{xy} + \frac{1}{4y^2}$
21. $\frac{1}{(z-4)} - \frac{1}{(z+4)}$
22. $\frac{x}{x-1} - \frac{1}{x+1}$
23. $\frac{1}{t^2+t} + \frac{1}{t^2-t}$
24. $\frac{1}{u^2-2u} - \frac{1}{u^2-4}$
25. $\frac{1}{x^2-1} - \frac{1}{(x-1)^2}$
26. $\frac{1}{y^2-y-2} + \frac{1}{y^2+y}$
27. $\frac{1}{s^2+2s+1} - \frac{1}{s^2-1}$
28. $\frac{1}{p^2-2p+1} - \frac{1}{p^2+p-2}$

B

29. $\frac{a+b}{a-b} + \frac{a-b}{a+b} + \frac{b-a}{a-b} + \frac{b-a}{a+b}$
30. $\frac{a+b}{a-b} + \frac{a-b}{a+b} - \frac{b-a}{a-b} + \frac{b-a}{a+b}$
31. $(x-y)^{-1} - (x+y)^{-1}$
32. $(x-y)^{-2} - (x+y)^{-2}$
33. $\frac{3}{x^2-5x+6} + \frac{2}{x^2-4}$
34. $\frac{1}{4t^2-4t+1} + \frac{1}{4t^2-1}$
35. $\frac{1}{2u^2-3uv+v^2} + \frac{1}{4u^2-v^2}$
36. $\frac{3}{4x^2-12xy+9y^2} + \frac{1}{2xy-3y^2}$
37. $\frac{x}{x-a} - \frac{x^2+a^2}{x^2-a^2} + \frac{a}{x+a}$
38. $\frac{3u}{2u-v} - \frac{2u}{2u+v} + \frac{2v^2}{4u^2-v^2}$
39. Prove: **a.** $\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}$ **b.** $\frac{a}{c} - \frac{b}{c} = \frac{a-b}{c}$

Find constants A and B that make the equation true.

C

40. $\frac{2x-9}{x^2-x-6} = \frac{A}{x-3} + \frac{B}{x+2}$
41. $\frac{x-7}{x^2+x-6} = \frac{A}{x+3} + \frac{B}{x-2}$

5-7 Complex Fractions

Objective To simplify complex fractions.

A fraction is a **complex fraction** if its numerator or denominator (or both) has one or more fractions or powers with negative exponents. For example,

$$\frac{\frac{7}{12}-\frac{1}{6}}{2+\frac{4}{9}} \quad \text{and} \quad \frac{x^{-1}-y^{-1}}{x^{-2}-y^{-2}}$$

are complex fractions. Here are two methods for simplifying such fractions.

Method 1 Simplify the numerator and denominator separately; then divide.

Method 2 Multiply the numerator and denominator by the LCD of all the fractions appearing in the numerator and denominator.

Example 1 Simplify $\dfrac{\frac{7}{12}-\frac{1}{6}}{2+\frac{4}{9}}$.

Solution

Method 1

$$\frac{\frac{7}{12}-\frac{1}{6}}{2+\frac{4}{9}} = \frac{\frac{7-2}{12}}{\frac{18+4}{9}} = \frac{\frac{5}{12}}{\frac{22}{9}} = \frac{5}{12}\cdot\frac{9}{22} = \frac{15}{88}$$

Method 2

$$\frac{\frac{7}{12}-\frac{1}{6}}{2+\frac{4}{9}} = \frac{\left(\frac{7}{12}-\frac{1}{6}\right)\cdot 36}{\left(2+\frac{4}{9}\right)\cdot 36} = \frac{21-6}{72+16} = \frac{15}{88}$$

Example 2 Simplify $\dfrac{z-\frac{1}{z}}{1-\frac{1}{z}}$.

Solution

Method 1

$$\frac{z-\frac{1}{z}}{1-\frac{1}{z}} = \frac{\frac{z^2-1}{z}}{\frac{z-1}{z}}$$

$$= \frac{z^2-1}{z}\cdot\frac{z}{z-1}$$

$$= \frac{\cancel{z}\cancel{(z-1)}(z+1)}{\cancel{z}\cancel{(z-1)}}$$

$= z+1$ ***Answer***

Method 2

$$\frac{z-\frac{1}{z}}{1-\frac{1}{z}} = \frac{\left(z-\frac{1}{z}\right)\cdot z}{\left(1-\frac{1}{z}\right)\cdot z}$$

$$= \frac{z^2-1}{z-1}$$

$$= \frac{\cancel{(z-1)}(z+1)}{\cancel{z-1}}$$

$= z+1$ ***Answer***

When the numerator or denominator of a complex fraction has powers with negative exponents, you should first rewrite the powers using positive exponents. Then simplify the fraction using either of the methods shown in Example 3.

Example 3 Simplify $\dfrac{a^{-1} - x^{-1}}{a^{-2} - x^{-2}}$.

Solution

Method 1

$$\frac{a^{-1} - x^{-1}}{a^{-2} - x^{-2}} = \left(\frac{1}{a} - \frac{1}{x}\right) \div \left(\frac{1}{a^2} - \frac{1}{x^2}\right)$$

$$= \frac{x - a}{ax} \div \frac{x^2 - a^2}{a^2x^2}$$

$$= \frac{x - a}{ax} \cdot \frac{a^2x^2}{x^2 - a^2}$$

$$= \frac{a^2x^2(x - a)}{ax(x + a)(x - a)}$$

$$= \frac{ax}{x + a}$$

Answer

Method 2

$$\frac{a^{-1} - x^{-1}}{a^{-2} - x^{-2}} = \frac{\left(\frac{1}{a} - \frac{1}{x}\right) \cdot a^2x^2}{\left(\frac{1}{a^2} - \frac{1}{x^2}\right) \cdot a^2x^2}$$

$$= \frac{ax^2 - a^2x}{x^2 - a^2}$$

$$= \frac{ax(x - a)}{(x + a)(x - a)}$$

$$= \frac{ax}{x + a}$$

Answer

Written Exercises

Simplify.

A **1.** $\dfrac{1 - \frac{1}{3}}{\frac{1}{2} - \frac{1}{6}}$ **2.** $\dfrac{\frac{1}{2} + \frac{1}{3}}{1 - \frac{1}{6}}$ **3.** $\dfrac{1 - \frac{4}{5}}{\frac{1}{4} - \frac{1}{5}}$

4. $\dfrac{\frac{2}{3} - \frac{5}{6}}{\frac{1}{3} + \frac{2}{9}}$ **5.** $\dfrac{x + 1}{1 + \frac{1}{x}}$ **6.** $\dfrac{z - \frac{1}{z}}{1 - \frac{1}{z}}$

7. $\dfrac{a - b}{a^{-1} - b^{-1}}$ **8.** $\dfrac{1 - xy^{-1}}{x^{-1} - y^{-1}}$ **9.** $\dfrac{u^{-2} - v^{-2}}{u^{-1} - v^{-1}}$

10. $\dfrac{a^{-2} - b^{-2}}{a^{-1} + b^{-1}}$ **11.** $\dfrac{\frac{1}{x^2} - \frac{1}{y^2}}{\frac{1}{x^2} + \frac{2}{xy} + \frac{1}{y^2}}$ **12.** $\dfrac{\frac{1}{p^2} - \frac{1}{q^2}}{\frac{2}{p^2} - \frac{1}{pq} - \frac{1}{q^2}}$

13. $\dfrac{h + h^{-2}}{1 + h^{-1}}$ **14.** $\dfrac{x^{-2} - x^2}{x^{-1} - x}$ **15.** $\dfrac{s^2 - t^{-2}}{s - t^{-1}}$

Simplify.

16. $\dfrac{\dfrac{1}{x}-\dfrac{1}{y}}{\dfrac{y}{x}-\dfrac{x}{y}}$

17. $\dfrac{\dfrac{2}{y+2}-1}{\dfrac{1}{y+2}+1}$

18. $\dfrac{1+\dfrac{1}{t-1}}{1-\dfrac{1}{t+1}}$

B 19. $\dfrac{\dfrac{1}{a+1}+\dfrac{1}{a-1}}{\dfrac{1}{a+1}-\dfrac{1}{a-1}}$

20. $\dfrac{\dfrac{1}{x}+\dfrac{1}{x+1}}{\dfrac{1}{x}-\dfrac{1}{x+1}}$

21. $\dfrac{1+\dfrac{1}{x-1}}{1+\dfrac{1}{x^2-1}}$

22. $\dfrac{\dfrac{1}{1-t}-\dfrac{1}{t}}{\dfrac{1}{1+t}-\dfrac{1}{t}}$

23. $\dfrac{\dfrac{a}{b}-\dfrac{a-b}{a+b}}{\dfrac{a}{b}+\dfrac{a+b}{a-b}}$

24. $\dfrac{\dfrac{u+v}{u-v}-\dfrac{u-v}{u+v}}{\dfrac{u+v}{u-v}+\dfrac{u-v}{u+v}}$

25. $\dfrac{1-\dfrac{2-\frac{1}{x}}{x}}{1-\dfrac{1}{x}}$

26. $\dfrac{u+\dfrac{1}{1+\frac{1}{u}}}{\dfrac{1}{u+1}}$

27. Evaluate to three decimal places. A calculator may be helpful.

a. $1+\dfrac{1}{2}$

b. $1+\dfrac{1}{2+\frac{1}{2}}$

c. $1+\dfrac{1}{2+\dfrac{1}{2+\frac{1}{2}}}$

d. $1+\dfrac{1}{2+\dfrac{1}{2+\dfrac{1}{2+\frac{1}{2}}}}$

(The farther this process is carried out, the closer the results will be to $\sqrt{2} = 1.41421 \ldots.$)

In Exercises 28–31, express $\dfrac{f(x+h)-f(x)}{h}$ as a single simplified fraction. (These exercises might be met in calculus.)

Sample $f(x) = \dfrac{1}{1-x}$

Solution

$$\frac{f(x+h)-f(x)}{h} = \frac{\dfrac{1}{1-(x+h)}-\dfrac{1}{1-x}}{h}$$
$$= \frac{(1-x)-(1-x-h)}{h(1-x-h)(1-x)}$$
$$= \frac{1}{(1-x-h)(1-x)}$$

C 28. $f(x) = \dfrac{1}{x}$

29. $f(x) = \dfrac{1}{x+1}$

30. $f(x) = \dfrac{1-x}{x}$

31. $f(x) = \dfrac{1}{x^2}$

In Exercises 32–35, express $f(f(x))$ as a single simplified fraction.

32. $f(x) = \dfrac{1}{x + 1}$

33. $f(x) = \dfrac{x}{x + 1}$

34. $f(x) = \dfrac{1 + x}{1 - x}$

35. $f(x) = (1 - x)^{-1}$

Mixed Review Exercises

Simplify.

1. $\dfrac{1}{x^2y} + \dfrac{1}{xy^2}$

2. $\dfrac{8}{a^2 - 3a} \cdot \dfrac{a^2 - 9}{6a}$

3. $\dfrac{36u^2}{25v} \div \dfrac{27u^4}{10v^3}$

4. $\dfrac{t}{t - 2} - \dfrac{1}{t + 2}$

5. $-\dfrac{8a^2b^3}{16a^3b^2}$

6. $\dfrac{x}{2y} + \dfrac{y}{2x}$

7. $\dfrac{x - 2}{4} - \dfrac{2 - x}{8}$

8. $\dfrac{2y - x}{x - 2y}$

9. $\dfrac{(a - 2)^2}{b^2} \div \dfrac{6a - 12}{4b}$

Find the unique solution of each system. Check your answer by using substitution.

10. $\begin{aligned} 4x - y &= 14 \\ 5x + 3y &= 9 \end{aligned}$

11. $\begin{aligned} 2x + 5y &= 17 \\ x - 2y &= -5 \end{aligned}$

12. $\begin{aligned} x - 4y &= -4 \\ 2x + 10y &= 1 \end{aligned}$

Self-Test 2

Vocabulary

rational algebraic expression (p. 227)
rational expression (p. 227)
rational function (p. 228)
least common denominator (p. 235)
complex fraction (p. 238)

Simplify.

1. $\dfrac{t^3 - t}{t^3 - t^2}$

2. $\dfrac{x^3 - x^2 - 4x + 4}{x^3 - 4x}$

Obj. 5-4, p. 227

3. $\dfrac{2p^4r^3}{3s^2} \cdot \dfrac{p^3r^2}{6s^3}$

4. $\dfrac{2u^2 + 3uv - 2v^2}{2u^2 - 3uv - 2v^2} \cdot \dfrac{2u + v}{2u - v}$

Obj. 5-5, p. 232

5. $\dfrac{1}{a^2} - \dfrac{2}{ab} + \dfrac{1}{b^2}$

6. $\dfrac{1}{4x^2 - 4x + 1} - \dfrac{1}{4x^2 - 1}$

Obj. 5-6, p. 235

7. $\dfrac{\frac{2}{3} - \frac{1}{4}}{1 - \frac{1}{6}}$

8. $\dfrac{1 - hk^{-1}}{h^{-1} - k^{-1}}$

Obj. 5-7, p. 238

Check your answers with those at the back of the book.

Problem Solving Using Fractional Equations

5-8 Fractional Coefficients

Objective To solve equations and inequalities having fractional coefficients.

To solve an equation or an inequality having fractional coefficients, it is helpful to multiply both sides of the open sentence by the least common denominator of the fractions.

Example 1 Solve $\frac{x^2}{2} = \frac{2x}{15} + \frac{1}{10}$.

Solution

$$\frac{x^2}{2} = \frac{2x}{15} + \frac{1}{10}$$

$$30 \cdot \frac{x^2}{2} = 30\left(\frac{2x}{15} + \frac{1}{10}\right)$$ Multiply both sides by 30, the LCM of 2, 15, and 10.

$$15x^2 = 4x + 3$$ Use the distributive property.

$$15x^2 - 4x - 3 = 0$$

$$(3x + 1)(5x - 3) = 0$$

$$x = -\frac{1}{3} \quad \text{or} \quad x = \frac{3}{5}$$

$\therefore$ the solution set is $\left\{-\frac{1}{3}, \frac{3}{5}\right\}$. ***Answer***

Example 2 Solve $\frac{x}{8} - \frac{x-2}{3} \geq \frac{x+1}{6} - 1$.

Solution

$$\frac{x}{8} - \frac{x-2}{3} \geq \frac{x+1}{6} - 1$$

$$24\left(\frac{x}{8} - \frac{x-2}{3}\right) \geq 24\left(\frac{x+1}{6} - 1\right)$$ Multiply both sides by 24, the LCM of 8, 3, and 6.

$$3x - 8x + 16 \geq 4x + 4 - 24$$ Use the distributive property.

$$-9x \geq -36$$

$$x \leq 4$$ Reverse the inequality.

$\therefore$ the solution set is $\{x: x \leq 4\}$. ***Answer***

Example 3 Crane A can unload the container ship in 10 h, and crane B can unload it in 14 h. Crane A started to unload the ship at noon and was joined by crane B at 2 P.M. At what time was the unloading job of the ship completed?

Solution

Step 1 Essentially, the problem asks for the number of hours crane A worked.

Step 2 Let $t =$ the number of hours crane A worked.

Then $t - 2 =$ the number of hours crane B worked.

The part of the job crane A can do in 1 hour $= \dfrac{1}{10}$.

Thus, the part of the job crane A can do in t hours is $t \cdot \dfrac{1}{10} = \dfrac{t}{10}$.

Similarly, the part of the job crane B can do in $t - 2$ hours is $(t - 2) \cdot \dfrac{1}{14} = \dfrac{t-2}{14}$.

Step 3

$$\underbrace{\text{Part done by crane A}}_{\frac{t}{10}} \quad \underset{+}{\text{plus}} \quad \underbrace{\text{part done by crane B}}_{\frac{t-2}{14}} \quad \underset{=}{\text{is}} \quad \underbrace{\text{the whole job.}}_{1}$$

Step 4 $\dfrac{t}{10} + \dfrac{t-2}{14} = 1$ Multiply both sides by the LCM, 70.

$$7t + 5t - 10 = 70$$
$$12t = 80$$
$$t = \frac{80}{12}$$
$$t = \frac{20}{3} = 6\frac{2}{3}$$

Step 5 The check is left for you.

$\therefore$ the unloading job was completed $6\frac{2}{3}$ h after noon, or at 6:40 P.M.

Recall that *percent* means hundredths. For example, 60% of 400 means 0.60×400, or 240. Fractions often enter problems through the use of percents, as in Example 4 at the top of the next page.

Example 4 A nurse wishes to obtain 800 mL of a 7% solution of boric acid by mixing 4% and 12% solutions. How much of each should be used?

Solution

Step 1 The problem asks for a number of mL of 4% and 12% solutions of boric acid to be used.

Step 2 Let x = number of mL of 4% solution.

Then $800 - x$ = number of mL of 12% solution.

Show the known facts in a table.

	mL of solution	× % boric acid	= mL of boric acid
4% solution	x	4%	$0.04x$
12% solution	$800 - x$	12%	$0.12(800 - x)$
Final mixture	800	7%	0.07(800)

Step 3 From the last column of the table, write an equation relating the 4% and 12% solutions to the final mixture.

$$0.04x + 0.12(800 - x) = 0.07(800)$$

or

$$4x + 12(800 - x) = 7(800)$$

To clear decimals, multiply both sides by 100.

Step 4

$$4x + 9600 - 12x = 5600$$
$$-8x = -4000$$
$$x = 500 \quad \text{(4\% solution)}$$
$$800 - x = 300 \quad \text{(12\% solution)}$$

Step 5 The check is left for you.

∴ the nurse should mix 500 mL of the 4% solution and 300 mL of the 12% solution. ***Answer***

Oral Exercises

Find an open sentence with integers as coefficients that is equivalent to the given open sentence.

1. $\frac{x}{2} - 1 = \frac{5}{6}$

2. $\frac{y}{6} + 2 = \frac{2}{3}$

3. $\frac{t}{3} - \frac{1}{2} > 1$

4. $\frac{x}{3} - \frac{2}{5} \le 2$

5. $u = \frac{2}{5} - \frac{u}{2}$

6. $\frac{t}{3} + \frac{5}{6} = \frac{3}{2}$

7. $2.7 + 0.5s = 1.2$

8. $0.14x - 0.06 = 0.15$

9. $\frac{z}{3} - \frac{3}{4} = \frac{5}{6}$

Written Exercises

Solve each open sentence.

A

1. $\frac{x}{9} + \frac{1}{6} = \frac{2}{3}$
2. $\frac{3u}{5} - \frac{5}{6} = \frac{u}{10}$
3. $\frac{2t - 1}{6} = \frac{t + 2}{4} + \frac{1}{3}$
4. $\frac{s - 2}{2} - \frac{s - 1}{5} = \frac{1}{4}$
5. $\frac{z}{4} - \frac{z - 1}{6} \leq \frac{5}{12}$
6. $\frac{t}{4} - \frac{t + 2}{3} + \frac{1}{6} \geq 0$
7. $\frac{r - 2}{8} < \frac{3r + 1}{6} + \frac{1}{3}$
8. $\frac{x - 5}{9} \leq \frac{2x}{15} - \frac{2}{5}$
9. $\frac{y^2}{4} - \frac{3y}{2} + 2 = 0$
10. $\frac{t^2}{6} - \frac{t}{2} - \frac{2}{3} = 0$
11. $\frac{z^2}{3} - \frac{z}{6} = 1$
12. $\frac{v^2}{6} + \frac{1}{8} = \frac{v}{3}$
13. $\frac{x^2}{9} + \frac{x - 1}{10} = 0$
14. $\frac{t(t - 1)}{3} = \frac{t + 1}{2}$
15. $\frac{y(2y - 1)}{2} = \frac{1 - y}{3}$
16. $\frac{w(w - 1)}{3} + \frac{1}{2} = \frac{w + 1}{4}$

B

17. $\frac{x(x + 1)}{5} - \frac{x + 1}{6} = \frac{1}{3}$
18. $\frac{2t(3t + 1)}{5} - \frac{t + 1}{2} = \frac{1}{10}$
19. $\frac{y(2y - 1)}{4} + \frac{3}{10} = \frac{y(y + 2)}{5}$
20. $\frac{u(u - 1)}{2} + \frac{1}{3} = \frac{u(2 - u)}{4}$
21. $\frac{y^2 + 4}{6} + \frac{y + 1}{3} < \frac{3}{2}$
22. $\frac{z^2 + 1}{6} \geq \frac{z + 2}{3}$
23. $\frac{t^2}{6} + \frac{t - 2}{4} \geq \frac{t + 1}{3}$
24. $\frac{u^2}{15} \leq \frac{u + 2}{6} - \frac{2}{5}$

Problems

In Problems 1–4, find the number described.

A

1. $\frac{5}{16}$ of $\frac{4}{5}$ of the number is 15.
2. 12 is $\frac{3}{5}$ of $\frac{10}{21}$ of the number.
3. 30 is 20% of 30% of the number.
4. 75% of 60% of the number is 36.
5. Pump A can unload the *Lunar Petro* in 30 h and pump B can unload it in 24 h. Because of an approaching storm, both pumps were used. How long did they take to empty the ship?
6. An old conveyor belt takes 21 h to move one day's coal output from the mine to a rail line. A new belt can do it in 15 h. How long does it take when both are used at the same time?
7. How much pure antifreeze must be added to 12 L of a 40% solution of antifreeze to obtain a 60% solution?
8. How much water must be evaporated from a 300 L tank of a 2% salt solution to obtain a 5% solution?

9. The river boat *Delta Duchess* paddled upstream at 12 km/h, stopped for 2 h of sightseeing, and paddled back at 18 km/h. How far upstream did the boat travel if the total time for the trip, including the stop, was 7 h?

10. Pam jogged up a hill at 6 km/h and then jogged back down at 10 km/h. How many kilometers did she travel in all if her total jogging time was 1 h 20 min?

11. The Computer Club invested $2200, part at 4.5% interest and the rest at 7%. The total annual interest earned was $144. How much was invested at each rate?

12. Lina Chen invested $24,000, part at 8% and the rest at 7.2%. How much did she invest at each rate if her income from the 8% investment is two thirds that of the 7.2% investment?

B 13. A pharmacist wishes to make 1.8 L of a 10% solution of boric acid by mixing 7.5% and 12% solutions. How much of each type of solution should be used?

14. How much of an 18% solution of sulfuric acid should be added to 360 mL of a 10% solution to obtain a 15% solution?

15. The county's new asphalt paving machine can surface 1 km of highway in 10 h. A much older machine can surface 1 km in 18 h. How long will it take them to surface 21 km of highway if they start at opposite ends and work day and night?

16. Pipes A and B can fill a storage tank in 8 h and 12 h, respectively. With the tank empty, pipe A was turned on at noon, and then pipe B was turned on at 1:30 P.M. At what time was the tank full?

17. Sharon drove for part of a 150 km trip at 45 km/h and the rest of the trip at 75 km/h. How far did she drive at each speed if the entire trip took her 2 h 40 min?

18. An elevator went from the bottom to the top of a tower at an average speed of 4 m/s, remained at the top for 90 s, and then returned to the bottom at 5 m/s. If the total elapsed time was $4\frac{1}{2}$ min, how high is the tower?

C 19. A commercial jet can fly from San Francisco to Dallas in 3 h. A private jet can make the same trip in $3\frac{1}{2}$ h. If the two planes leave San Francisco at noon, after how many hours is the private jet twice as far from Dallas as the commercial jet?

20. A car radiator is filled with 5 L of a 25% antifreeze solution. How many liters must be drawn off and replaced by a 75% antifreeze solution to leave the radiator filled with a 55% antifreeze solution?

21. The rail line between two cities consists of two segments, one 96 km longer than the other. A passenger train averages 60 km/h over the shorter segment, 120 km/h over the longer, and 100 km/h for the entire trip. How far apart are the cities?

5-9 Fractional Equations

Objective To solve and use fractional equations.

An equation in which a variable occurs in a denominator, such as

$$\frac{3}{x^2 - 7x + 10} + 2 = \frac{x - 4}{x - 5},$$

is called a **fractional equation.** When you solve a fractional equation by multiplying both sides by the LCD of the fractions, the resulting equation is *not* always equivalent to the original one.

Example 1 Solve $\frac{3}{x^2 - 7x + 10} + 2 = \frac{x - 4}{x - 5}$.

Solution $x^2 - 7x + 10 = (x - 2)(x - 5)$. So the LCD is $(x - 2)(x - 5)$.

$$(x - 2)(x - 5)\left[\frac{3}{x^2 - 7x + 10} + 2\right] = (x - 2)(x - 5)\left[\frac{x - 4}{x - 5}\right]$$

$$3 + 2(x - 2)(x - 5) = (x - 2)(x - 4)$$

$$3 + 2x^2 - 14x + 20 = (x - 2)(x - 4)$$

$$2x^2 - 14x + 23 = x^2 - 6x + 8$$

$$x^2 - 8x + 15 = 0$$

$$(x - 3)(x - 5) = 0$$

$$x = 3 \quad \text{or} \quad x = 5$$

Check: When x equals 2 or 5, a denominator in the original equation has a value of 0. Therefore, 2 and 5 are not in the domain of x and cannot be solutions. Since 5 is not permissible, check 3 in the original equation:

$$\frac{3}{3^2 - 7 \cdot 3 + 10} + 2 \stackrel{?}{=} \frac{3 - 4}{3 - 5}$$

$$-\frac{3}{2} + 2 \stackrel{?}{=} \frac{-1}{-2}$$

$$\frac{1}{2} = \frac{1}{2} \quad \checkmark$$

$\therefore$ the solution set is $\{3\}$. ***Answer***

In Example 1, the equation obtained by multiplying the given equation by $(x - 2)(x - 5)$ has the *extraneous root* 5. An **extraneous root** is a root of the transformed equation that is *not* a root of the original equation.

Caution: If you transform an equation by multiplying by a polynomial, always check each root of the new equation in the *original* one.

Example 2 One pump can empty the town swimming pool in 7 h less time than a smaller second pump can. Together they can empty the pool in 12 h. How much time would it take the larger pump alone to empty it?

Solution

Step 1 The problem asks for the number of hours for the larger pump to empty the pool.

Step 2 Let $t =$ the number of hours for the larger pump to empty the pool.
Then $t + 7 =$ the number of hours for the smaller pump to empty the pool.

Now write an expression for the part of the pool that is emptied by each pump in 1 h and the part emptied by each in 12 h.

$$\frac{1}{t} = \text{part of pool emptied by larger pump in 1 h}$$

$$\frac{12}{t} = \text{part of pool emptied by larger pump in 12 h}$$

$$\frac{1}{t+7} = \text{part of pool emptied by smaller pump in 1 h}$$

$$\frac{12}{t+7} = \text{part of pool emptied by smaller pump in 12 h}$$

Step 3 The sum of the parts emptied by each pump in 12 h is 1.

$$\frac{12}{t} + \frac{12}{t+7} = 1$$

Step 4

$$12(t + 7) + 12t = t(t + 7)$$
$$t^2 - 17t - 84 = 0$$
$$(t + 4)(t - 21) = 0$$
$$t = -4 \quad \text{or} \quad t = 21$$

Since the time t cannot be negative, reject $t = -4$ as an answer.

Step 5 *Check* $t = 21$: If the larger pump can empty the pool in 21 h, then the smaller can empty it in 21 + 7, or 28 h. Therefore, the parts of the pool emptied by the pumps in 12 h are $\frac{12}{21} = \frac{4}{7}$ and $\frac{12}{28} = \frac{3}{7}$.

$$\frac{4}{7} + \frac{3}{7} = 1 \quad \checkmark$$

∴ the larger pump can empty the pool in 21 h.
Answer

Example 3 Because of strong headwinds, an airplane's ground speed (see page 131) for the first half of a 2000 km trip averaged only 600 km/h. What must its ground speed be for the rest of the trip if it is to average 720 km/h for the entire trip?

Solution Let r = the ground speed in km/h that the airplane must travel for the second half of its trip. Use the fact that $time = \frac{distance}{rate}$ and make a table.

	Distance (km)	Rate (km/h)	Time (h)
First half	1000	600	$\frac{1000}{600} = \frac{5}{3}$
Second half	1000	r	$\frac{1000}{r}$
Entire trip	2000	720	$\frac{2000}{720} = \frac{25}{9}$

$$\underbrace{\frac{5}{3}}_{\text{Time for first half of trip}} \overset{\text{plus}}{+} \underbrace{\frac{1000}{r}}_{\text{Time for second half of trip}} \overset{\text{equals}}{=} \underbrace{\frac{25}{9}}_{\text{Time for entire trip}}$$

Multiply both sides by $9r$: $\quad 15r + 9000 = 25r$

$$r = 900$$

The check is left for you.

$\therefore$ the ground speed for the second half must be 900 km/h. ***Answer***

Written Exercises

Solve and check. If an equation has no solution, say so.

A

1. $\frac{3}{t} - \frac{1}{3t} = \frac{2}{3}$

2. $\frac{2}{x} + \frac{1}{4} = \frac{1}{x}$

3. $\frac{1}{x} = \frac{2}{x-3}$

4. $\frac{3}{u+2} = \frac{1}{u-2}$

5. $\frac{2}{s+3} - \frac{1}{s-3} = 0$

6. $\frac{5}{2r+1} - \frac{3}{2r-1} = 0$

7. $\frac{t}{t-1} = \frac{t+2}{t}$

8. $\frac{x-3}{x} = \frac{x-4}{x-2}$

9. $\frac{6t^2 - t - 1}{3(t^2+1)} = 2$

10. $\frac{(y+1)^2}{(y-3)^2} = 1$

11. $\frac{x}{x+3} + \frac{1}{x-3} = 1$

12. $\frac{1}{s} + \frac{s}{s+2} = 1$

13. $\frac{1}{y-2} + \frac{1}{y+2} = \frac{4}{y^2-4}$

14. $\frac{k}{k+1} + \frac{k}{k-2} = 2$

15. $\frac{3}{x+1} - \frac{1}{x-2} = \frac{1}{x^2-x-2}$

16. $\frac{1}{t-1} + \frac{1}{t+2} = \frac{3}{t^2+t-2}$

17. $\frac{6}{t} - \frac{2}{t-1} = 1$

18. $\frac{1}{x+1} - \frac{1}{x+2} = \frac{1}{2}$

B 19. $\dfrac{u}{u-2}+\dfrac{30}{u+2}=9$

20. $\dfrac{1}{s+3}+\dfrac{1}{s-5}=\dfrac{1-s}{s+3}$

21. $\dfrac{2}{x-1}-\dfrac{x}{x+3}=\dfrac{6}{x^2+2x-3}$

22. $\dfrac{x}{x+3}+\dfrac{1}{x-1}=\dfrac{4}{x^2+2x-3}$

23. $\dfrac{5}{u^2+u-6}=2-\dfrac{u-3}{u-2}$

24. $\dfrac{y}{y-2}-\dfrac{2}{y+3}=\dfrac{10}{y^2+y-6}$

25. $\dfrac{t}{t-1}=\dfrac{1}{t+2}+\dfrac{3}{t^2+t-2}$

26. $\dfrac{9}{t^2-2t-8}+\dfrac{t}{t+2}=2$

27. $\left(\dfrac{x-3}{x+1}\right)^2=2\cdot\dfrac{x-3}{x+1}+3$

28. $\left(\dfrac{t+3}{t-1}\right)^2=2+\dfrac{t+3}{t-1}$

C 29. $\dfrac{\dfrac{1}{x^2}-x^2}{\dfrac{1}{x}+x}=\dfrac{3}{2}$

30. $\left(\dfrac{1}{x}-\dfrac{x}{2}\right)\left(\dfrac{2}{x}-x\right)=\dfrac{1}{2}$

Problems

Solve.

A 1. Find two positive numbers that differ by 8 and whose reciprocals differ by $\frac{1}{6}$.

2. Find two numbers whose sum is 25 and the sum of whose reciprocals is $\frac{1}{6}$.

$\frac{1}{2}+\frac{2}{1}$

3. The reciprocal of half a number increased by half the reciprocal of the number is $\frac{1}{2}$. Find the number.

4. The reciprocal of one third of a number decreased by one third of the reciprocal of the number is $\frac{1}{3}$. Find the number.

5. A town's old street sweeper can clean the streets in 60 h. The old sweeper together with a new sweeper can clean the streets in 15 h. How long would it take the new sweeper to do the job alone?

6. The intake pipe can fill a certain tank in 6 h when the outlet pipe is closed, but with the outlet pipe open it takes 9 h. How long would it take the outlet pipe to empty a full tank?

7. During 60 mi of city driving, Jenna averaged 15 mi/gal. She then drove 140 mi on an expressway and averaged 25 mi/gal for the entire 200 mi. Find the average fuel consumption on the expressway.

8. Helped by a strong jet stream, a Los Angeles-to-Boston plane flew 10% faster than usual and made the 4400 km trip in 30 min less time than usual. At what speed does the plane usually fly?

9. The excursion boat *Holiday* travels 35 km upstream and then back again in 4 h 48 min. If the speed of the *Holiday* in still water is 15 km/h, what is the speed of the current?

10. Tim paddled his kayak 12 km upstream against a 3 km/h current and back again in 5 h 20 min. In that time how far could he have paddled in still water?

11. Members of the Computer Club were assessed equal amounts to raise $1200 to buy some software. When 8 new members joined, the per-member assessment was reduced by $7.50. What was the new size of the club?

12. Members of the Ski Club contributed equally to obtain $1800 for a holiday trip. When 6 members found that they could not go, their contributions were refunded and each remaining member then had to pay $10 more to raise the $1800. How many went on the trip?

13. To measure the speed of the jet stream, a weather plane left its base at noon and flew 800 km directly against the stream with an air speed of 750 km/h. It then returned directly to its base, arriving at 2:24 P.M. What was the speed of the jet stream?

14. When Ace Airlines changed to planes that flew 100 km/h faster than its old ones, the time of its 2800 km Dallas-Seattle flight was reduced by 30 min. Find the speed of the new planes.

B 15. Elvin drove halfway from Ashton to Dover at 40 mi/h and the rest of the way at 60 mi/h. What was his average speed for the whole trip? (*Hint:* Let the distance for the whole trip be, say, 100 mi.)

16. Elizabeth drove the first half of a trip at 36 mi/h. At what speed should she cover the remaining half in order to average 45 mi/h for the whole trip? (See the hint in Exercise 15.)

17. A train averaged 120 km/h for the first two thirds of a trip and 100 km/h for the whole trip. Find its average speed for the last third of the trip.

18. Because of traffic Maria could average only 40 km/h for the first 20% of her trip, but she averaged 75 km/h for the whole trip. What was her average speed for the last 80% of her trip?

19. Pipe A can fill a tank in 5 h. Pipe B can fill it in 2 h less time than it takes pipe C, a drainpipe, to empty the tank. With all three pipes open, it takes 3 h to fill the tank. How long would it take pipe C to empty it?

20. An elevator went from the bottom to the top of a 240 m tower, remained there for 12 s, and returned to the bottom in an elapsed time of 2 min. If the elevator traveled 1 m/s faster on the way down, find its speed going up.

A number x is the **harmonic mean** of a and b if $\frac{1}{x}$ is the average of $\frac{1}{a}$ and $\frac{1}{b}$.

C **21.** The harmonic mean of a number and 5 is 8. Find the number.

22. Find two positive numbers that differ by 12 and have harmonic mean 5.

23. Suppose that a vehicle averages u km/h for the first half of a trip and v km/h for the second half. Show that its average speed for the whole trip is the harmonic mean of u and v.

Mixed Review Exercises

Simplify.

1. $\dfrac{x^2 - 1}{1 - x}$ **2.** $\dfrac{8t^2 - 2}{8t + 4}$ **3.** $\dfrac{32m^5n}{24m^2n^2}$

4. $\dfrac{(4z^3)^{-2}}{(2z^4)^{-1}}$ **5.** $\dfrac{a^{-1} + b^{-2}}{a^{-2} + b^{-1}}$ **6.** $\dfrac{c^2 - c - 12}{c^2 - 2c - 8}$

Solve.

7. $\dfrac{2x - 1}{3} = \dfrac{3x + 2}{6}$ **8.** $3 - \dfrac{1}{2}x \le 4$ **9.** $\dfrac{x^2}{6} = \dfrac{x}{3} + \dfrac{5}{2}$

10. $\dfrac{3x}{4} + 1 > \dfrac{x - 1}{2}$ **11.** $\dfrac{3}{2}x - 1 = 5$ **12.** $\dfrac{1}{2}x^2 + \dfrac{7}{3}x - 4 < 0$

Self-Test 3

Vocabulary fractional equation (p. 247) extraneous root (p. 247)

Solve.

1. $\dfrac{3t - 2}{8} = \dfrac{t - 1}{12} + 1$ **2.** $\dfrac{y^2}{3} + \dfrac{y}{15} - \dfrac{2}{5} = 0$ **Obj. 5-8, p. 242**

3. One computer can process a payroll in 6 h. An older computer can do the same job in 7 h 30 min. How long would it take both computers to do the processing?

Solve and check. If an equation has no solution, say so.

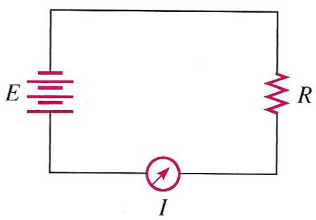
4. $\dfrac{x}{x - 1} = \dfrac{x - 2}{x - 3}$ **5.** $\dfrac{z}{z - 2} - \dfrac{1}{z + 3} = \dfrac{10}{z^2 + z - 6}$ **Obj. 5-9, p. 247**

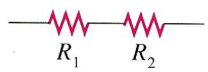
6. A train averaged 80 km/h for the first half of its trip. How fast must it travel for the second half of the trip in order to average 96 km/h for the whole trip?

Compare your answers with those at the back of the book.

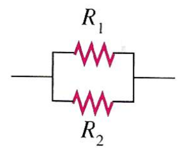
Application / Electrical Circuits

Figure (a) pictures an electrical circuit containing an E-volt (V) battery, an R-ohm (Ω) resistance, and an ammeter for measuring the current I in amperes (A). Ohm's law states that $I = \frac{E}{R}$.





$R_c = R_1 + R_2$

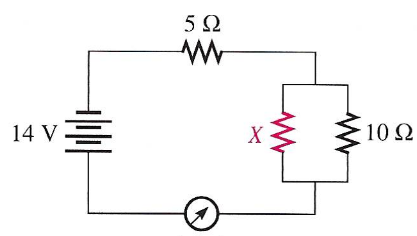


$\frac{1}{R_c} = \frac{1}{R_1} + \frac{1}{R_2}$

(a) (b) (c)

Figures (b) and (c) show resistances R_1 and R_2 in series and in parallel, respectively, and also formulas for finding the total resistance R_c of the combination.

Example Find the quantity labeled X in the following circuit.



Solution The resistance of the 10 ohm–X ohm combination is R_c, where

$$\frac{1}{R_c} = \frac{1}{10} + \frac{1}{X} = \frac{X + 10}{10X}.$$

$$\therefore R_c = \frac{10X}{X + 10}.$$

The total resistance is

$$\begin{aligned} R &= 5 + R_c \\ &= 5 + \frac{10X}{X + 10}. \end{aligned}$$

From $I = \frac{E}{R}$, or $IR = E$, you have

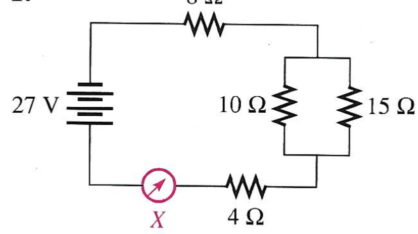
$$1.2\left(5 + \frac{10X}{X + 10}\right) = 14$$

Solving the equation, you find that $X = 20\ \Omega$. ***Answer***

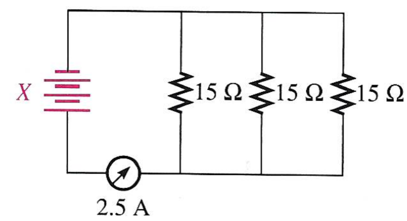
Exercises

Find the quantity labeled X in each of the following circuits.

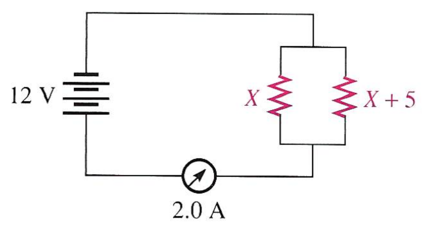
1.



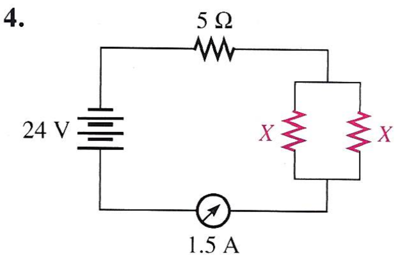
2.



3.



4.



Chapter Summary

1. The definitions

$$a^0 = 1 \quad \text{and} \quad a^{-n} = \frac{1}{a^n}$$

permit zero and negative integers as exponents. The laws of exponents stated on page 212 continue to hold.

2. In *scientific notation* a positive number is expressed in the form

$$m \times 10^n$$

where $1 \le m < 10$ and n is an integer. The digits of m should all be *significant*.

3. A quotient of polynomials is called a *rational expression*. Such an expression is simplified if its numerator and denominator have no common factors other than 1.

4. To multiply and divide rational expressions, you can use the product and quotient rules for fractions:

$$\frac{p}{q} \cdot \frac{r}{s} = \frac{pr}{qs}$$

and

$$\frac{p}{q} \div \frac{r}{s} = \frac{p}{q} \cdot \frac{s}{r}$$

5. To add or subtract rational expressions, find their *least common denominator* (LCD), express each as an equivalent fraction having the LCD as denominator, and then add or subtract.

6. To simplify a *complex fraction,* use either of these methods:

   Simplify the numerator and denominator separately; then divide.

   Multiply the numerator and denominator by the LCD of all the fractions appearing in the numerator and the denominator.

7. To solve an equation or an inequality having fractional coefficients, multiply both sides by the LCD of the fractions. To solve a *fractional equation* (one having a variable in the denominator), use the same method, but always check the results for any extraneous roots that may have been introduced.

Chapter Review

Give the letter of the correct answer.

1. Simplify $\frac{18x^3y}{12x^2y^4}$. 5-1

a. $\frac{2}{3xy^3}$ **b.** $\frac{3x}{2y^3}$ **c.** $\frac{2xy^3}{3}$ **d.** $\frac{3y^3}{2x}$

2. Simplify $\frac{(a^2bc^3)^3}{(-ab^3c^2)^2}$.

a. $-\frac{a^4c^5}{b^3}$ **b.** $a^4b^3c^5$ **c.** $-\frac{ac}{b^2}$ **d.** $\frac{a^4c^5}{b^3}$

3. Simplify $\frac{(p^2q)^{-1}}{p^2q^{-1}}$. 5-2

a. $\frac{1}{p^4}$ **b.** q^2 **c.** $\frac{q^2}{p^4}$ **d.** 1

4. Simplify $\left(\frac{m^2}{n}\right)^{-2} \cdot \left(\frac{m^0}{n^{-1}}\right)^3$.

a. $\frac{n^5}{m}$ **b.** m^4n **c.** $\frac{1}{m^4n}$ **d.** $\frac{n^5}{m^4}$

5. Express 0.000543 in scientific notation. 5-3

a. 5.43×10^{-3} **b.** 5.43×10^{-4} **c.** 54.3×10^{-5} **d.** 543×10^{-6}

6. Find a one-significant-digit estimate for $(0.0593)(2045) \div 0.32$.

a. 5×10^3 **b.** 4×10^2 **c.** 2×10^{-1} **d.** 3×10

7. Simplify $\frac{y^2 - 4}{y^2 + y - 6}$. 5-4

a. $\frac{2}{y + 3}$ **b.** $\frac{y - 2}{y - 3}$ **c.** $\frac{y - 4}{y - 6}$ **d.** $\frac{y + 2}{y + 3}$

8. Find the zeros of $f(x) = \frac{2x^2 - x - 1}{2x^2 + 3x - 2}$.

a. $-\frac{1}{2}, 1$ b. $\frac{1}{2}, -2$ c. $\frac{1}{2}, -1$ d. $-\frac{1}{2}, 2$

9. Simplify $\frac{u^2 + 3u}{u^2 + 2u - 3} \cdot \frac{u^3 - u}{u^2 - u - 2}$. **5-5**

a. $\frac{u^2}{u - 2}$ b. $u - 2$ c. $\frac{1}{u - 2}$ d. $\frac{u}{u - 2}$

10. Simplify $\frac{m^2}{n^3} \cdot \frac{n}{m^4} \div \frac{1}{mn}$.

a. mn b. m^3n^3 c. $\frac{1}{mn}$ d. $\frac{1}{m^3n^3}$

11. Simplify $\frac{a + 2}{2a} + \frac{1 - a}{a^2}$. **5-6**

a. $\frac{a^2 + 2}{2a^3}$ b. $\frac{a^2 + 2}{2a^2}$ c. $\frac{a + 2}{2a}$ d. $\frac{a + 1}{a}$

12. Simplify $\frac{x - 1}{x + 1} - \frac{x^2 - x - 6}{x^2 + 4x + 3}$.

a. $\frac{1}{x + 1}$ b. $\frac{2x^2 + x - 9}{(x + 1)(x + 3)}$

c. $\frac{3}{x + 3}$ d. $\frac{x - 9}{(x + 1)(x + 3)}$

13. Simplify $\dfrac{y - \frac{2}{y + 1}}{1 - \frac{2}{y + 1}}$. **5-7**

a. $y + 1$ b. $\frac{y}{y - 1}$ c. $\frac{y - 2}{y + 1}$ d. $y + 2$

14. Solve $\frac{x + 1}{6} < x - \frac{3x - 2}{4}$. **5-8**

a. $\{x: x > -4\}$ b. $\{x: x < 4\}$ c. $\{x: x < -4\}$ d. $\{x: x > 4\}$

15. Solve $\frac{x(x - 2)}{6} = \frac{x + 1}{8}$.

a. $\{3\}$ b. $\left\{-3, \frac{1}{4}\right\}$ c. $\left\{-\frac{1}{4}, 3\right\}$ d. $\{-1, 0, 2\}$

16. How much of an 8% saline solution should be added to 600 mL of a 3% solution to produce a 5% solution?

a. 200 mL b. 300 mL c. 400 mL d. 500 mL

17. Solve $\frac{2}{x + 2} + \frac{x^2}{x^2 - 4} = \frac{1}{x - 2}$. **5-9**

a. $\{-3, 2\}$ b. $\{-3\}$ c. $\{-2, 3\}$ d. $\{3\}$

18. Bill rowed 30 km upstream against a 2 km/h current and back again in a total of 8 h. How fast can he row in still water?

a. 7 km/h b. 8 km/h c. 9 km/h d. 10 km/h

Chapter Test

Simplify. Use only positive exponents in the answer.

1. $\dfrac{(3xy^3)^2}{6x^5y^4}$

2. $\dfrac{m^4}{4n^3}\left(\dfrac{2n}{m^3}\right)^3$ 5-1

3. $pq^4(p^{-1}q^2)^{-2}$

4. $\left(\dfrac{u^2}{v}\right)^{-2}\left(\dfrac{v^3}{u}\right)^{-1}$ 5-2

Evaluate. Give each answer in scientific notation with two significant digits.

5. $(2.1 \times 10^3)(1.6 \times 10^{-5})$

6. $\dfrac{5.2 \times 10^2}{1.2 \times 10^{-4}}$ 5-3

Find (a) the domain and (b) the zeros of each function.

7. $f(x) = \dfrac{3x - 2}{x^2 - x - 12}$

8. $f(x) = \dfrac{x^2 - x - 12}{3x - 2}$ 5-4

Simplify.

9. $\dfrac{6x^3}{y^2} \div \dfrac{3xy}{2}$

10. $\dfrac{u^2 - 9}{u + 2} \div \dfrac{u + 3}{u^2 - 4}$ 5-5

11. $\dfrac{p - 2}{2p} + \dfrac{p + 3}{3p}$

12. $\dfrac{1}{s - 1} - \dfrac{2}{s^2 - 1}$ 5-6

13. $\dfrac{ab^{-1} + 1}{a^{-1}b + 1}$

14. $\dfrac{\dfrac{x + y}{x - y}}{\dfrac{1}{x} - \dfrac{1}{y}}$ 5-7

Solve.

15. $\dfrac{x + 7}{8} - \dfrac{1}{2} \geq \dfrac{x}{4}$

16. $\dfrac{x^2}{9} = \dfrac{x + 2}{2}$ 5-8

17. One crew can detassel a field of corn in 6 days. Another crew can do the same job in 4 days. If the slower crew works alone for the first two days of detasseling and then is joined by the faster crew, how long will it take the two crews to finish?

18. Solve $\dfrac{3}{x^2 - 2x - 8} + \dfrac{1}{x + 2} = \dfrac{4}{x^2 - 16}$. 5-9

19. The members of a computer club were to be assessed equally to raise \$400 for the purchase of software. When four more persons joined the club, the per-member assessment was reduced by \$5.00. What was the new size of the club membership?