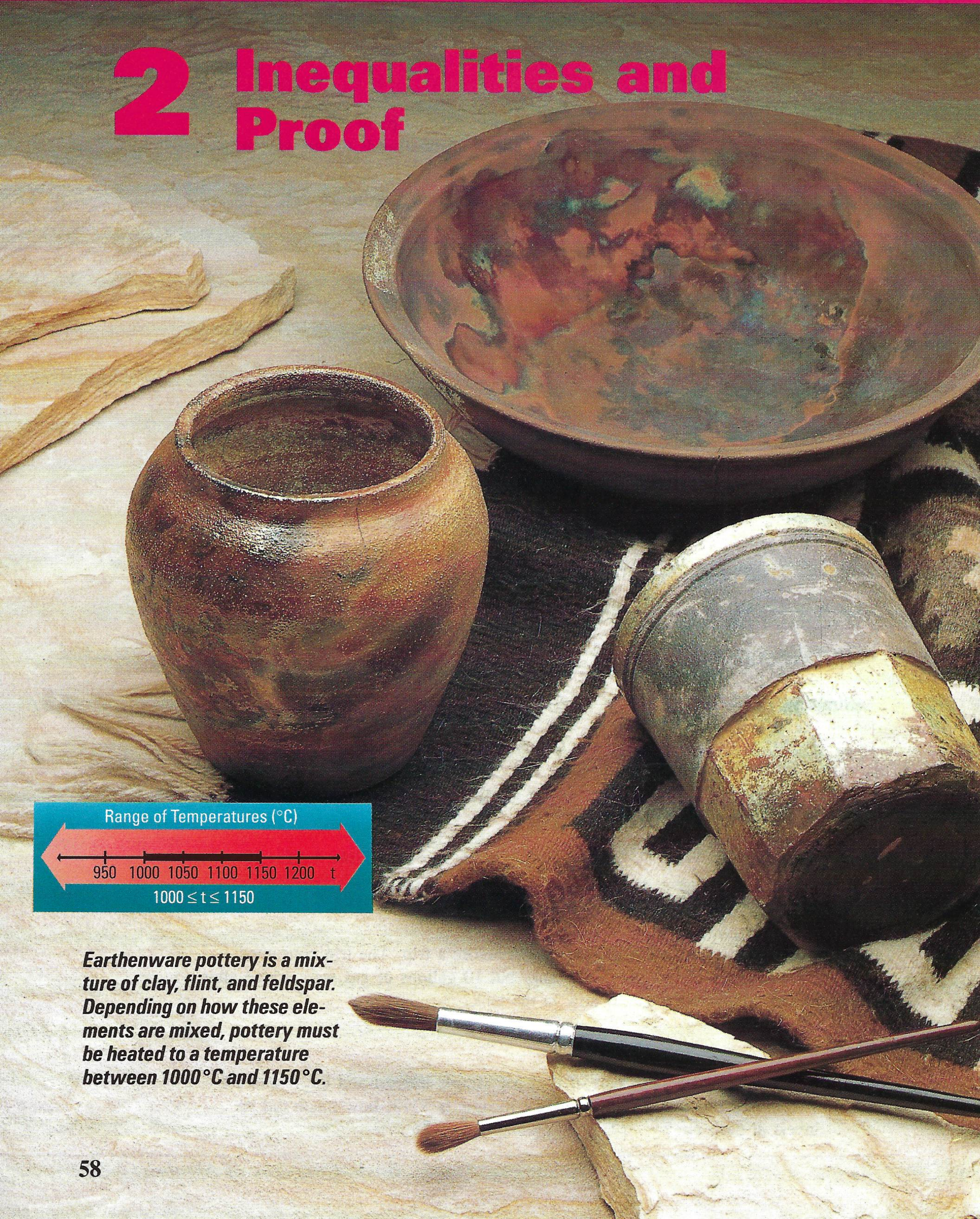
2 Inequalities and Proof



Earthenware pottery is a mixture of clay, flint, and feldspar. Depending on how these elements are mixed, pottery must be heated to a temperature between 1000°C and 1150°C.

Working with Inequalities

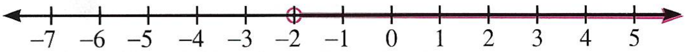
2-1 Solving Inequalities in One Variable

Objective To solve simple inequalities in one variable.

The inequality

$$x > -2$$

is satisfied by every real number greater than -2. The graph of this inequality is shown in red on the number line below. Note the use of the open circle to show that -2 is not a solution.



To find solutions of more complicated inequalities like

$$5x + 17 < 2$$

and

$$5(3 - t) < 7 - t$$

you use methods similar to those used to solve equations. These methods are based on the properties of order for real numbers stated below. Recall that a real number c is called *positive* if $c > 0$ and *negative* if $c < 0$.

Properties of Order

Let a, b, and c be any real numbers.

Comparison Property

Exactly one of the following statements is true:

$$a < b, \quad a = b, \quad \text{or } a > b.$$

Transitive Property

If $a < b$ and $b < c$, then $a < c$.

Addition Property

If $a < b$, then $a + c < b + c$.

Multiplication Property

1. If $a < b$ and c is *positive*, then $ac < bc$.
2. If $a < b$ and c is *negative*, then $ac > bc$.

Since subtraction is defined in terms of addition, the addition property of order applies to subtraction. Similarly, the multiplication properties apply to division. Furthermore, since the statement $b > a$ has the same meaning as $a < b$, the properties of order hold true if $<$ and $>$ are interchanged throughout.

When you multiply or divide both sides of an inequality by a *negative* number, you must *reverse* the direction of the inequality. For example:

$$5 < 8, \quad \text{but} \quad (-2)5 > (-2)8 \qquad (\text{that is, } -10 > -16)$$

$$1 > -4, \quad \text{but} \quad \frac{1}{-2} < \frac{-4}{-2} \qquad \left(\text{that is, } -\frac{1}{2} < 2\right)$$

You solve an inequality by transforming it into an inequality whose solution set is easy to see. Transformations that produce **equivalent inequalities,** that is, inequalities with the same solution set, are listed in the chart below.

Transformations that Produce Equivalent Inequalities

1. Simplifying either side of an inequality.
2. Adding to (or subtracting from) each side of an inequality the same number or the same expression.
3. Multiplying (or dividing) each side of an inequality by the same *positive* number.
4. Multiplying (or dividing) each side of an inequality by the same *negative* number and *reversing* the direction of the inequality.

Notice how these transformations are used in the following example.

Example 1 Solve each inequality and graph its solution set.

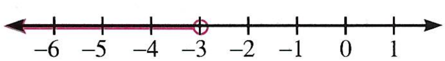
a. $5x + 17 < 2$ **b.** $5(3 - t) < 7 - t$

Solution

a.

$$5x + 17 - 17 < 2 - 17$$
$$5x < -15$$
$$\frac{5x}{5} < \frac{-15}{5}$$
$$x < -3$$

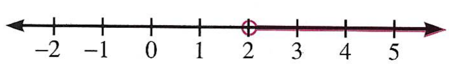
$\therefore$ the solution set consists of all real numbers less than -3.



b.

$$15 - 5t < 7 - t$$
$$15 - 5t + t < 7 - t + t$$
$$15 - 4t < 7$$
$$-4t < -8$$
$$\frac{-4t}{-4} > \frac{-8}{-4}$$
$$t > 2$$

$\therefore$ the solution set consists of all real numbers greater than 2.



To quickly check whether the direction of the inequality in your answer is correct, choose a test point and see if it satisfies the given inequality. For example, in part (a) of Example 1, substituting 0 for x gives $17 < 2$, which is false. So 0 is not in the solution set and the direction of the inequality $x < -3$ is correct.

The solution set of part (a) of Example 1 can be written as $\{x: x < -3\}$, which is read, "the set of all x such that x is less than -3." Similarly, the solution set of part (b) can be written $\{t: t > 2\}$. This notation will be used throughout the rest of this book.

Some inequalities are true for all real numbers, and others have no solution, as Example 2 illustrates.

Example 2 Solve each inequality and graph its solution set.

a. $4x > 2(3 + 2x)$

b. $2t < -\frac{1}{2}(-6 - 4t)$

Solution

a. $4x > 2(3 + 2x)$

$4x > 6 + 4x$

$0 > 6$

Since the equivalent inequality $0 > 6$ is *false,* the given inequality is false and has no solution.

$\therefore$ the solution set is $\emptyset$, and there is no graph.

b. $2t < -\frac{1}{2}(-6 - 4t)$

$2t < 3 + 2t$

$0 < 3$

Since $0 < 3$ is *true,* the given inequality is true for all values of t.

$\therefore$ the solution set is {real numbers}.

Oral Exercises

Explain how to transform the first inequality into the second inequality.

Sample $-3t > 15;\ t < -5$

Solution Divide each side by -3 and reverse the direction of the inequality.

1. $x - 3 < 4;\ x < 7$

2. $3s > 6;\ s > 2$

3. $-\frac{u}{4} > 3;\ u < -12$

4. $k + 2 < 1;\ k < -1$

5. $2(y + 1) + 3 < y;\ 2y + 5 < y$

6. $10 > -5y;\ -2 < y$

7. $\frac{3x + 1}{2} < 5;\ 3x + 1 < 10$

8. $5t < 3(t - 2) + 1;\ 5t < 3t - 5$

9. $z - 3 > 3z - 7;\ -3 > 2z - 7$

10. $4x - 7 > x - 2;\ 4x > x + 5$

Match each inequality with the graph of its solution set.

11. $-6x > 3$

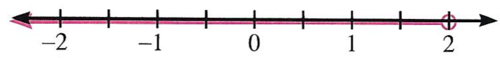
12. $4 - x > 2$

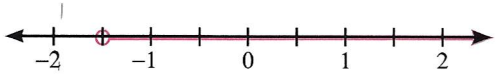
13. $1 - x < x$

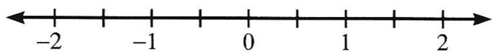
14. $2(x - 1) < 2x - 3$

15. $2(x + 1) + 1 > 0$

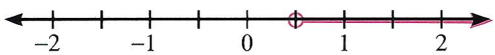
16. $2(x - 1) > 2x - 3$

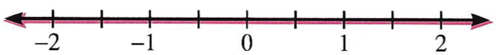
a.


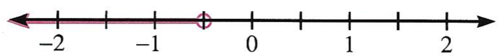
b.


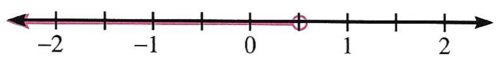
c.


d.


e.


f.


g.


h.


17. a. Explain the meaning of this statement: *If* $a < 0$, *then* $a^2 > 0$.

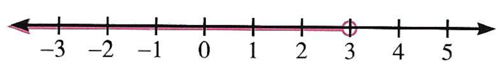
b. Decide whether the statement in (a) is true or false. Give a convincing argument to support your answer.

Written Exercises

Solve each inequality and graph each solution set that is not empty.

Sample $1 - 2t > -5$

Solution $\{t: t < 3\}$



A

1. $x - 7 > -5$

2. $y + 4 < 3$

3. $2t < 6$

4. $3u > -6$

5. $-5x < 10$

6. $-12 > -4y$

7. $-\frac{t}{2} > \frac{3}{2}$

8. $-\frac{3}{4}k < -6$

9. $3s - 1 > -4$

10. $2r + 5 < -1$

11. $y < 7y - 24$

12. $3t > 6t + 12$

13. $2 - h < 4 + h$

14. $1 + 2x < 2(x - 1)$

15. $5(2u + 3) > 2(u - 3) + u$

16. $3(x - 2) - 2 < x - 5$

17. $5(x - 7) + 2(1 - x) > 3(x - 11)$

18. $4s + 3(2 - 3s) < 5(2 - s)$

19. $7y - 2(y - 4) > 6 - (2 - y)$

20. $4(2 - x) - 3(1 + x) < 5(1 - x)$

Solve.

B 21. $k - 3(2 - 4k) < 7 - (8k - 9 + k)$

22. $\frac{2}{3}t - (2 - 3t) < 5t + 2(1 - t)$

23. $4(y + 2) - 9y > y - 3(2y + 1) - 1$

24. $4[5x - (3x - 7)] < 2(4x - 5)$

Tell whether each statement is true for all real numbers. If you think it is not, give a numerical example to support your answer.

25. If $a < b$, then $a - c < b - c$.

26. If $a < b$, then $a - b < 0$.

27. If $a < b$, then $a^2 < b^2$.

28. If $a < b$, then $a^3 < b^3$.

29. If $a < b$ and $c < d$, then $a + c < b + d$.

30. If $a < b$ and $c < d$, then $a - c < b - d$.

C 31. If $a \neq b$, then $a^2 + b^2 > 2ab$.

32. If $a > 0$ and $a \neq 1$, then $a + \frac{1}{a} > 2$.

33. The following statement is true:

If $a > 0$ and $a < 1$, then $a^2 < a$.

Write a short paragraph explaining the meaning of this statement to someone who has studied arithmetic but not algebra.

Mixed Review Exercises

Simplify.

1. $(8t - 5) - (5 - 8t)$

2. $(-4)(5)(-1)(-3)$

3. $(-2)^4(-p)^3$

4. $\frac{5^2 - 7^2}{5 - 7}$

5. $\frac{8cd - 6}{-2}$

6. $\frac{6(2 + 3)}{6 \cdot 2 + 3}$

7. $|3(-4) - 2|$

8. $-2(5 - 8)^3$

9. $2a^2 - 5a - (a^2 - 7)$

10. $6x - y + 2x - 4y$

11. $|-4| - |-9|$

12. $4(3 - m) - (2m + 1)$

Computer Exercises

For students with some programming experience.

1. Write a computer program that will print all solutions of the inequality $ax + b < c$, where a, b, and c are entered by the user. In this program the domain of x will be a set of consecutive integers, where the smallest and largest members of the set will also be entered by the user. (*Hint:* Use a FOR . . . NEXT loop to test each integer from smallest to largest.)
2. Run the program in Exercise 1 to find solutions of the inequality $3x - 7 < 17$ for the following domains.

   a. $\{1, 2, \ldots, 6\}$ **b.** $\{5, 6, \ldots, 12\}$ **c.** $\{10, 11, \ldots, 20\}$
3. Modify the program in Exercise 1 so that if there are no solutions in the given domain, the computer will print a message to this effect.
4. Run the program in Exercise 3 to find solutions of the inequality $5x + 3 < 12$ for the following domains.

   a. $\{6, 7, \ldots, 12\}$ **b.** $\{-10, -9, \ldots, 6\}$ **c.** $\{1, 2, \ldots, 6\}$

Career Note / *Automotive Engineer*

When you read that automobile manufacturers have increased the fuel efficiency of cars, you are actually reading about the work of automotive engineers. Automotive engineering is a specialized field of mechanical engineering, which is concerned with the use, production, and transmission of mechanical power.

The role of an automotive engineer, as with all engineers, is to apply the theories and principles of science and mathematics to the solution of practical problems. Automotive engineers in particular are responsible for designing automobiles that deliver efficient and economical performance. Evaluating the overall cost, reliability, and safety of automobiles is also part of an automotive engineer's work.

To assist them in their work, automotive engineers rely upon calculators, computer simulations, and other engineers.

Computer-aided design systems have become especially important for the production and analysis of automotive engine-and-body designs.

2-2 Solving Combined Inequalities

Objective To solve conjunctions and disjunctions.

The general admission price at the Cinema V theater is \$4.50. Children 12 years of age or under and adults who are at least 65 are charged only half price. Therefore, you must pay the full price of \$4.50 if your age, a, satisfies both of the inequalities $a > 12$ and $a < 65$. The combined inequality "$a > 12$ and $a < 65$" is an example of a *conjunction*.

A sentence formed by joining two sentences with the word *and* is called a **conjunction.** A conjunction is true when *both* sentences are true.

Example 1 Graph the solution set of the conjunction $x > -2$ *and* $x < 3$.

Solution The conjunction is true for all values of x between -2 and 3. The graph is shown below. Notice that the numbers *between* -2 and 3 include neither -2 nor 3.

The conjunction "$x > a$ and $x < b$" is usually written as

$$a < x < b,$$

which is read "x is greater than a and less than b." Using this notation, the solution set of Example 1 is written $\{x: -2 < x < 3\}$.

A sentence formed by joining two sentences with the word *or* is called a **disjunction.** A disjunction is true when *at least one* of the sentences is true.

You learned earlier that "$x \leq 2$" means "x is less than or equal to 2" and therefore represents the disjunction

$$x < 2 \quad or \quad x = 2.$$

In the graph of this disjunction, shown below, a solid red dot has been used to show that 2 is included in the solution set.

The chart below summarizes how to solve conjunctions and disjunctions of open sentences in one variable.

Conjunction: Find the values of the variable for which *both* sentences are true.

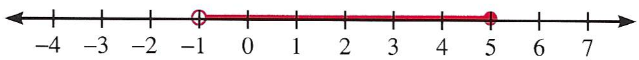
Disjunction: Find the values of the variable for which *at least one* of the sentences is true.

Example 2 Solve $3 < 2x + 5 \leq 15$ and graph its solution set.

Solution 1 You can first rewrite the inequality using *and*. Then solve both inequalities.

$$\begin{array}{ccc} 3 < 2x + 5 & \text{and} & 2x + 5 \leq 15 \\ -2 < 2x & \downarrow & 2x \leq 10 \\ -1 < x & \text{and} & x \leq 5 \end{array}$$

$\therefore$ the solution set is $\{x\text{: } -1 < x \leq 5\}$. ***Answer***



Solution 2 You can solve the inequality using this shortened method that involves operating on all three parts of the inequality at the same time.

$$\begin{array}{ll} 3 < 2x + 5 \leq 15 & \text{Subtract 5 from all three parts.} \\ -2 < 2x \leq 10 & \text{Divide all three parts by 2.} \\ -1 < x \leq 5 & \end{array}$$

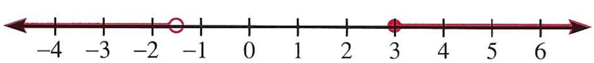
$\therefore$ the solution set is $\{x\text{: } -1 < x \leq 5\}$. ***Answer***

Example 3 Solve the disjunction $7 - 2y \leq 1$ *or* $3y + 10 < 4 - y$ and graph its solution set.

Solution

$$\begin{array}{ccc} 7 - 2y \leq 1 & \text{or} & 3y + 10 < 4 - y \\ -2y \leq -6 & \downarrow & 4y < -6 \\ y \geq 3 & \text{or} & y < -\frac{3}{2} \end{array}$$

$\therefore$ the solution set is $\left\{y\text{: } y \geq 3 \text{ or } y < -\frac{3}{2}\right\}$. ***Answer***



Oral Exercises

Tell whether each conjunction or disjunction is true or false.

1. $-5 < -3$ and $3 < 5$

2. $-1 < 2$ and $-1 < -2$

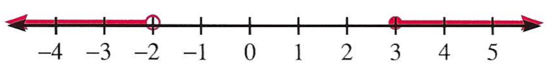
3. $-6 > -2$ or $-6 > 2$

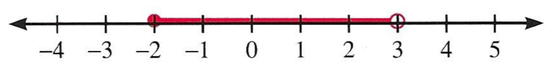
4. $-6 > -3$ or $-3 < 6$

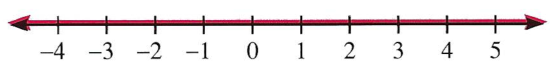
5. $-7 < -5$ or $7 < -5$

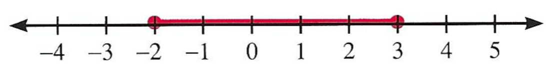
6. $-4 < -6$ and $-4 < 6$

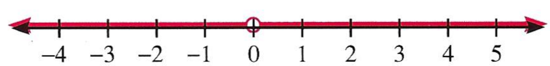
Match each graph with one of the open sentences in a–h.

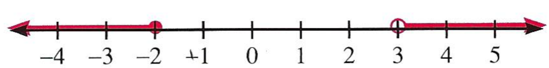
7. 

8. 

9. 

10. 

11. 

12. 

a. $-2 \le x \le 3$

b. $-2 < x < 3$

c. $x < 0$ or $x > 0$

d. $x \ge -2$ or $x \le 3$

e. $x > 3$ or $x \le -2$

f. $-2 \le x < 3$

g. $x < -2$ or $x \ge 3$

h. $x \le -2$ or $x > 3$

13. The inequality $x \ne 2$ is equivalent to the combined inequality $x < 2$ __?__ $x > 2$.

14. Explain why it would be incorrect to write $-2 < x < -5$.

Written Exercises

Solve each conjunction or disjunction and graph each solution set that is not empty.

A

1. $3 \le x < 5$

2. $z > -1$ and $z < 3$

3. $t < 1$ or $t \ge 3$

4. $p > 1$ or $p < 1$

5. $y \ge -1$ and $y \ge 3$

6. $y \ge -1$ or $y \ge 3$

7. $t > 0$ or $t < 2$

8. $w < 0$ and $w \ge 4$

9. $0 \le x - 2 < 3$

10. $2 > y + 2 \ge 0$

11. $-1 > 2r - 5 > -9$

12. $-1 \le 3z + 2 \le 8$

13. $2z - 1 \le 5$ or $3z - 5 > 10$

14. $3k + 7 < 1$ or $2k - 3 > 1$

15. $2t + 7 \geq 13$ or $5t - 4 < 6$

16. $2x + 3 > 1$ or $5x - 9 \leq 6$

17. $2t + 7 \geq 13$ and $5t - 4 < 6$

18. $2x + 3 > 1$ and $5x - 9 \leq 6$

19. $-5 < 1 - 2k < 3$

20. $-6 \leq 2 - 3m \leq 7$

21. $-3 < 2 - \frac{d}{3} \leq -1$

22. $3 \geq 1 - \frac{n}{2} > -2$

B 23. $7q - 1 > q + 11$ or $-11q > -33$

24. $5n - 1 > 0$ and $4n + 2 < 0$

25. $x - 7 < 3x - 5 < x + 11$

26. $3y + 5 \geq 2y + 1 > y - 1$

27. $-\frac{3}{4}m \geq m - 1$ or $-\frac{3}{4}m < m + 1$

28. $3z + 7 \leq 4z$ and $3z + 7 > -4z$

Solve.

29. $-3 \leq -2(t - 3) < 6$

30. $-5 < 2(2 - s) + 1 \leq 9$

31. $\frac{t}{4} + 2 < t + 3$ and $t - 3 > \frac{t}{2} - 4$

32. $\frac{r - 3}{6} \leq r - 1$ or $\frac{r - 6}{3} \leq r + 4$

C 33. $0 < 1 - x \leq 3$ or $-1 \leq 2x - 3 \leq 5$

34. $1 < -(2s + 1) < 5$ or $1 < 2s - 1 < 5$

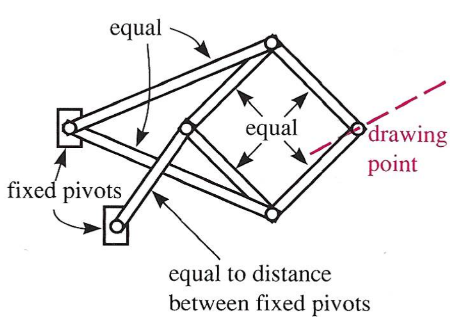
35. $2 < \frac{y + 6}{2} < 5$ and $(4 - y > 5$ or $4 + y > 7)$

36. $\left(x \leq \frac{x + 4}{3} + 2 \text{ or } x \geq 2x - 1\right)$ and $1 \leq \frac{x - 1}{2} \leq 3$

Historical Note / *Linkages*

The problem of constructing a linkage made of hinged rods that will draw a straight line has practical as well as theoretical interest. James Watt, the inventor of the steam engine, tried to construct such a device to guide the motion of the engine's piston. Watt's linkage, however, drew only an approximation of a straight line.

In 1864 a French army officer named Peaucellier solved the problem with the linkage shown at the right, in which one pivot moves in a perfectly straight line. Peaucellier was awarded the mechanical prize of the Institute of France in 1873.



2-3 Problem Solving Using Inequalities

Objective To solve word problems by using inequalities in one variable.

Sometimes solving a word problem involves using an inequality.

Example 1 A bus is to be chartered for the senior class trip. The basic fare is \$9.50 per passenger. If more than 20 people go, everyone's fare is reduced by \$.30 for each passenger over this number (20). *At least* how many people must go to make the fare less than \$7.50 per passenger?

Solution

Step 1 The problem asks for the least number of passengers needed to make the fare for each less than \$7.50.

Step 2 Let n = the number of passengers.

Then $n - 20$ = the number of passengers over 20;

$0.30(n - 20)$ = the amount each passenger's fare is reduced; and

$9.50 - 0.30(n - 20)$ = the reduced fare per passenger.

Step 3 Reduced fare per passenger is less than \$7.50.

$$9.50 - 0.30(n - 20) < 7.50$$

Step 4 Multiplying both sides of the inequality in Step 3 by 10 clears the decimals and gives this equivalent inequality to solve:

$$95 - 3(n - 20) < 75$$
$$95 - 3n + 60 < 75$$
$$-3n < -80$$
$$n > \frac{-80}{-3}$$
$$n > 26\frac{2}{3}$$

Interpret the result: Since the number of passengers must be an integer, $n \geq 27$.

Step 5 *Check:* Is the reduced fare less than \$7.50? If at least 27 people go, the fare per passenger is reduced by $0.30(27 - 20) = 2.10$.
Then the reduced fare is $9.50 - 2.10 = 7.40 < 7.50$.

Is 27 the least number of passengers? Try 26. If 26 people go, the fare per passenger is reduced by $0.30(26 - 20) = 1.80$.
Then the reduced fare is $9.50 - 1.80 = 7.70 > 7.50$.

$\therefore$ at least 27 passengers must go. ***Answer***

Certain phrases can be translated into mathematical terms using inequalities. Here is a list of the more common ones.

Phrase	Translation
x is at least a. x is no less than a.	$x \geq a$
x is at most b. x is no greater than b.	$x \leq b$
x is between a and b. x is between a and b, inclusive.	$a < x < b$ $a \leq x \leq b$

Example 2 Find all sets of 4 consecutive integers whose sum is between 10 and 20.

Solution

Step 1 The problem asks for 4 consecutive integers; their sum must be greater than 10 and less than 20.

Step 2 Let $n =$ the first of these integers.
Then the other three are $n + 1$, $n + 2$, and $n + 3$.

Step 3 $10 < \quad \text{the sum} \quad < 20$

$$10 < n + (n + 1) + (n + 2) + (n + 3) < 20$$

Step 4

$$10 < 4n + 6 < 20$$
$$4 < 4n < 14$$
$$1 < n < \frac{14}{4}$$

Interpret the result: Since n is an integer, there are only two values possible for n: 2 and 3. There are two sets of consecutive integers that fulfill the requirements of the problem:

$$\{2, 3, 4, 5\} \text{ and } \{3, 4, 5, 6\}$$

Step 5 *Check:* Is the sum between 10 and 20?

For $\{2, 3, 4, 5\}$: $10 < 2 + 3 + 4 + 5 < 20$
$10 < 14 < 20$ ✓

For $\{3, 4, 5, 6\}$: $10 < 3 + 4 + 5 + 6 < 20$
$10 < 18 < 20$ ✓

To complete the check, you must show that any other set of four consecutive integers will not satisfy the requirements. In fact, you need only eliminate the set of "next greater" integers, $\{4, 5, 6, 7\}$, and the set of "next smaller" integers, $\{1, 2, 3, 4\}$. That work is left for you to do.

$\therefore$ the required sets are $\{2, 3, 4, 5\}$ and $\{3, 4, 5, 6\}$. ***Answer***

Problems

Solve.

A 1. For the Hawks' 80 basketball games next year, you can buy separate tickets for each game at \$9 each, or you can buy a season ticket for \$580. At most how many games could you attend at the \$9 price before spending more than the cost of a season ticket?

2. The usual toll charge to use the Bingham tunnel is 50 cents. If you purchase a special sticker for \$5.50, the toll is only 35 cents. At least how many trips through the tunnel are needed before the sticker costs less than paying for each trip separately?

3. The length of a rectangle is 5 cm more than twice its width. Find the largest possible width if the perimeter is at most 64 cm.

4. The lengths of the legs of an isosceles triangle are integers. The base is half as long as each leg. What are the possible lengths of the legs if the perimeter is between 6 units and 16 units?

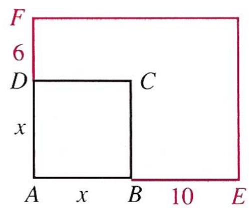
5. Find all sets of three consecutive *odd* integers whose sum is between 20 and 30.

6. Find all sets of three consecutive *even* integers whose sum is between 25 and 45.

7. Jeannie's scores on her first four tests were 80, 65, 87, and 75. What will she have to score on her next test to obtain an average of at least 80 for the term?

8. Jim's second test score was 8 points higher than his first score. His third score was 88. He had a B average (between 80 and 89, inclusive) for the three tests. What can you conclude about his first test score?

9. The sides $\overline{AB}$ and $\overline{AD}$ of a square are extended 10 cm and 6 cm, respectively, to form sides $\overline{AE}$ and $\overline{AF}$ of a rectangle. At most how long is the side of the square if the perimeter of the rectangle is at least twice the perimeter of the square?



10. The three sides of an equilateral triangle are increased by 20 cm, 30 cm, and 40 cm, respectively. The perimeter of the resulting triangle is between twice and three times the perimeter of the original triangle. What can you conclude about the length of a side of the original triangle?

B 11. The telephone company offers two types of service. With Plan A, you can make an unlimited number of local calls per month for \$18.50. With Plan B, you pay \$6.50 monthly, plus 10 cents for each min of calls after the first 40 min. At least how many min would you have to use the telephone each month to make Plan A the better option?

12. During the first 20 mi of a 50 mi bicycle race, Roger's average speed was 16 mi/h. What must his average speed be during the remainder of the race if he is to finish the race in less than 2.5h?

13. A subway train makes six stops of equal length during its 21 km run. The train is actually moving for 20 min of the trip. At most how long can the train remain at each station if the average speed for the trip, including stops, is to be at least 36 km/h?

14. The length of a rectangular sheet of paper was twice its width. After 1 cm was trimmed from each edge of the sheet, the perimeter was at most 1 m. Find the largest possible dimensions of the trimmed sheet.

C 15. A swimming pool is 5 m longer than it is wide and is surrounded by a deck 2 m wide. The area of the pool and deck together is at least 140 m^2 greater than the area of the pool alone. What can you conclude about the dimensions of the pool?

16. Find all triples of consecutive integers such that 11 times the largest of the integers is at least 46 more than the product of the other two.

Mixed Review Exercises

Solve each open sentence and graph each solution set that is not empty.

1. $3x - 2 \geq -8$
2. $\frac{1}{2}t \leq -2$ or $t - 4 \geq -3$
3. $2(m - 2) > 4 - 3(1 - m)$
4. $-2y < -6$ and $y + 3 \leq 1$
5. $7 - 4d < 3$
6. $-1 < 5 - 2p < 5$

Evaluate if $x = -7$ and $y = 3$.

7. $|x + y|$
8. $|x| + |y|$
9. $|xy|$
10. $|x| \cdot |y|$

Self-Test 1

Vocabulary equivalent inequalities (p. 60) disjunction (p. 65) conjunction (p. 65)

Solve each inequality and graph the solution set.

1. $6m - 13 \leq 5$ 2. $2y + 9 < 5(y + 3)$ 3. $8 - 3x > -7$ **Obj. 2-1, p. 59**

4. $\frac{n - 3}{2} \geq n - 1$ 5. $1 < 4a + 5 < 9$ 6. $c - 3 > 2c$ or $\frac{c}{3} \geq 1$ **Obj. 2-2, p. 65**

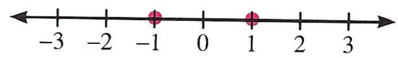
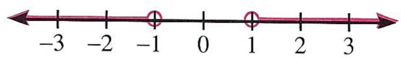
7. Bill's scores on his first two tests were 75 and 82. What will he have to score on his next test to obtain an average of at least 80? **Obj. 2-3, p. 69**

Working with Absolute Value

2-4 Absolute Value in Open Sentences

Objective To solve open sentences involving absolute value.

If you think of the absolute value of a real number x (see page 3) as the distance between the graph of x and the origin on a number line, you can see why the sentences below are equivalent.

Sentence	Equivalent Sentence	Graph
$\lvert x\rvert = 1$ Distance between x and 0 equals 1.	$x = -1$ or $x = 1$	
$\lvert x\rvert > 1$ Distance between x and 0 is greater than 1.	$x < -1$ or $x > 1$	
$\lvert x\rvert < 1$ Distance between x and 0 is less than 1.	$-1 < x < 1$	

You can often solve an open sentence involving absolute value by first writing an equivalent disjunction or conjunction.

Example 1 Solve $|3x - 2| = 8$.

Solution $|3x - 2| = 8$ is equivalent to this disjunction:

$$\begin{array}{rcr} 3x - 2 = -8 & \text{or} & 3x - 2 = 8 \\ 3x = -6 & \downarrow & 3x = 10 \\ x = -2 & \text{or} & x = \frac{10}{3} \end{array}$$

$\therefore$ the solution set is $\left\{-2, \frac{10}{3}\right\}$. ***Answer***

Example 2 Solve $|3 - 2t| < 5$.

Solution $|3 - 2t| < 5$ is equivalent to this conjunction:

$$\begin{array}{rcl} -5 < & 3 - 2t & < 5 \\ -8 < & -2t & < 2 \\ 4 > & t & > -1 \end{array}$$

$\therefore$ the solution set is $\{t\colon -1 < t < 4\}$. ***Answer***

Example 3 Solve $|2z - 1| + 3 \geq 8$ and graph its solution set.

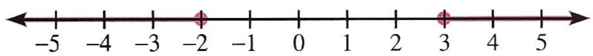
Solution First transform the inequality to an equivalent inequality in which the expression involving absolute value is alone on one side.

$$|2z - 1| + 3 \geq 8$$
$$|2z - 1| + 3 - 3 \geq 8 - 3$$
$$|2z - 1| \geq 5$$

The last inequality is equivalent to this disjunction:

$$\begin{array}{rcr} 2z - 1 \leq -5 & \text{or} & 2z - 1 \geq 5 \\ 2z \leq -4 & \downarrow & 2z \geq 6 \\ z \leq -2 & \text{or} & z \geq 3 \end{array}$$

$\therefore$ the solution set is $\{z\colon z \leq -2 \text{ or } z \geq 3\}$. ***Answer***



You can help guard against errors by testing one value from each region of the graph. Substitute values in the original inequality $|2z - 1| + 3 \geq 8$.

Try $z = -4$:	$|2(-4) - 1| + 3 = |-9| + 3 = 12 \geq 8$	True	√
Try $z = 0$:	$|2 \cdot 0 - 1| + 3 = |-1| + 3 = 4 \geq 8$	False	√
Try $z = 4$:	$|2 \cdot 4 - 1| + 3 = |7| + 3 = 10 \geq 8$	True	√

You can tell at a glance that an inequality such as $|x - 3| \geq -2$ is true for all real numbers x, because the absolute value of every real number is nonnegative. On the other hand, an inequality such as $|t + 5| < -1$ has $\emptyset$ as its solution set (why?).

Oral Exercises

Express each open sentence as an equivalent conjunction or disjunction without absolute value.

Sample 1 $|3t - 1| > 2$ **Solution** $3t - 1 < -2$ or $3t - 1 > 2$

1. $|x| \leq 3$ **2.** $|t| = 2$ **3.** $|z| > 0$ **4.** $|y - 3| \leq 2$

5. $|s + 3| = 3$ **6.** $|2x - 3| \geq 1$ **7.** $|3t - 1| \leq 2$ **8.** $|5 - 2z| < 3$

Express each conjunction or disjunction as an equivalent open sentence involving absolute value.

Sample 2 $-1 \leq x - 2 \leq 1$ **Solution** $|x - 2| \leq 1$

9. $u = -3$ or $u = 3$ **10.** $t \geq -3$ and $t \leq 3$ **11.** $3 > 4(x - 1) > -3$

Written Exercises

A **1–8.** Graph the solution set of each open sentence in Oral Exercises 1–8.

Solve and graph the solution set.

9. $|2t + 5| < 3$

10. $|3x + 2| > 4$

11. $|2u - 5| = 0$

12. $8 = |5y + 2|$

13. $\left|1 - \frac{x}{3}\right| \geq \frac{2}{3}$

14. $\left|1 - \frac{p}{2}\right| \leq 2$

15. $0 \leq |4u - 7|$

16. $|3r - 12| > 0$

17. $\left|\frac{t - 2}{4}\right| \leq \frac{1}{2}$

18. $1 > |2 - 0.8n|$

Solve.

B **19.** $|x + 5| - 3 = 1$

20. $|2t - 3| + 2 = 5$

21. $|2u - 1| + 3 \leq 6$

22. $4 - |3k + 1| < 2$

23. $7 - 3|4d - 7| \geq 4$

24. $6 + 5|2r - 3| \geq 4$

25. $4 + 2\left|\frac{3t - 5}{2}\right| > 5$

26. $2\left|\frac{2t - 5}{3}\right| - 3 \geq 5$

27. $7 + 5|c| \leq 1 - 3|c|$

28. $\frac{1}{2}|d| + 5 \geq 2|d| - 13$

Graph the solution set of each open sentence.

C **29.** $2 < |w| < 4$

30. $1 \leq |s - 2| \leq 3$

31. $1 \leq |2x + 1| < 3$

32. $0 < |2 - r| \leq 2$

Solve.

33. $|2x| \leq |x - 3|$

34. $|t| > |2t - 6|$

Computer Exercises

For students with some programming experience.

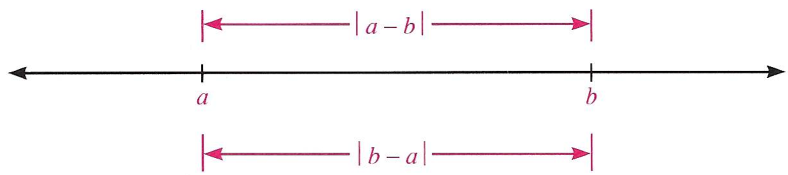
1. Write a program to list all integers x in the interval $-50 \leq x \leq 50$ that are solutions of an open sentence of the form $a < |cx + d| < b$. The values of a, b, c, and d are to be entered by the user. If no integers in the given interval satisfy the inequality, have the output state this. You will need to use the BASIC function ABS in your program.
2. Use the program in Exercise 1 to find the integer solutions of each open sentence.

   a. $17 < |3x - 25| < 35$ **b.** $1 < |18x + 120| < 100$

2-5 Solving Absolute Value Sentences Graphically

Objective To use number lines to obtain quick solutions to certain equations and inequalities involving absolute value.

You know that on a number line the distance between the graph of a real number x and the origin is $|x|$. *The distance on the number line between the graphs of real numbers a and b is $|a - b|$, or equivalently, $|b - a|$.* You can use this fact to solve many open sentences almost at sight.



Example 1 Find the distance between the graphs of each pair of numbers.

a. 8 and 17 **b.** −11 and −6 **c.** −9 and 12

Solution

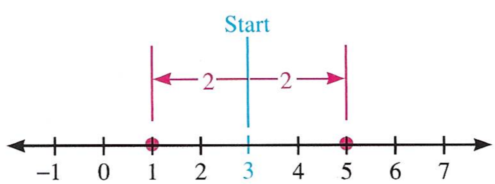
a. $|8 - 17| = |-9| = 9$ ***Answer***

b. $|-11 - (-6)| = |-11 + 6| = |-5| = 5$ ***Answer***

c. $|-9 - 12| = |-21| = 21$ ***Answer***

Example 2 Solve $|x - 3| = 2$.

Solution To satisfy $|x - 3| = 2$, x must be a number whose distance from 3 is 2 units. So, to find x, start at 3 and move 2 units in each direction on a number line.



You arrive at 1 and 5 as the values of x.

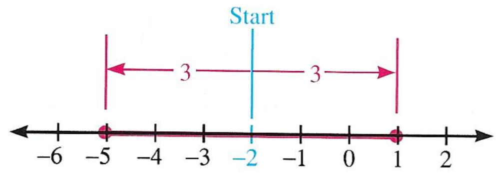
$\therefore$ the solution set is $\{1, 5\}$. ***Answer***

Example 3 Solve $|y + 2| \le 3$.

Solution $y + 2 = y - (-2)$

Therefore, $|y + 2| \le 3$ is equivalent to $|y - (-2)| \le 3$.

So the distance between y and -2 must be 3 units or less. To find y, start at -2 and move 3 units in each direction on a number line.



The numbers up to and including 1 and the numbers down to and including -5 will satisfy the inequality.

$\therefore$ the solution set is $\{y: -5 \leq y \leq 1\}$. ***Answer***

Certain equations and inequalities, such as the ones in Examples 1–3, lend themselves more easily to a graphic solution. With these types, the expression involving absolute value is of the form $|x - \text{constant}|$. When the expression is more complicated, as in Example 4, a graphic method can also be applied, though not as easily. For these types, you might prefer to use the algebraic method learned in the previous lesson.

Example 4 Solve $|5 - 2t| > 3$.

Solution Use the facts that $|a - b| = |b - a|$ and $|ab| = |a| \cdot |b|$ (page 27) to rewrite $|5 - 2t|$ this way:

$$|5 - 2t| = |2t - 5| = \left|2 \cdot \left(t - \frac{5}{2}\right)\right|$$
$$= |2| \cdot \left|t - \frac{5}{2}\right|$$
$$= 2\left|t - \frac{5}{2}\right|$$

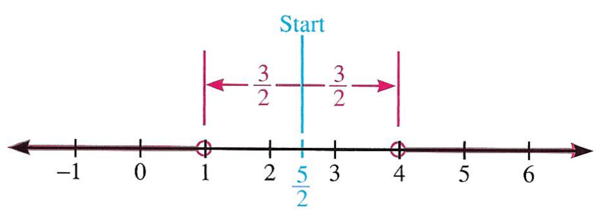
Therefore, the given inequality

$$|5 - 2t| > 3$$

is equivalent to

$$2\left|t - \frac{5}{2}\right| > 3, \text{ or } \left|t - \frac{5}{2}\right| > \frac{3}{2}$$

To find t, start at $\frac{5}{2}$ and move *more than* $\frac{3}{2}$ units to the right and to the left.



$\therefore$ the solution set is $\{t: t < 1 \text{ or } t > 4\}$. ***Answer***

Oral Exercises

Find the distance between the graphs of each pair of numbers.

1. 3 and -4
2. -6 and -1
3. 12 and 5
4. -7 and 1
5. 11 and 8
6. -11 and 8
7. 11 and -8
8. -11 and -8
9. t and $4t$ if **(a)** t is positive, and **(b)** t is negative.
10. $-2k$ and $2k$ if **(a)** k is positive, and **(b)** k is negative.

Translate each open sentence into a statement involving distance.

Sample $|z + 2| \geq 3$ **Solution** The distance between z and -2 is at least 3.

11. $|t| \leq 5$
12. $|r - 2| > 1$
13. $|3 - s| = 4$
14. $|x + 1| \leq 4$
15. $|u + 5| \geq 3$
16. $|2 - y| < 2$

Translate each statement into an open sentence using absolute value.

Sample The numbers whose distance from -3 is at most 2 **Solution** $|x - (-3)| \leq 2$, or $|x + 3| \leq 2$

17. The numbers whose distance from 2 is less than 5
18. The numbers whose distance from -3 is at least 3
19. The numbers whose distance from -1 is not more than 3
20. The numbers whose distance from 4 is equal to 4
21. The numbers whose distance from $-\frac{2}{3}$ is not less than 5
22. The numbers whose distance from 0 is at most 6
23. The numbers whose distance from -1.2 is greater than 4.8
24. The numbers whose distance from a is not more than h

Written Exercises

Solve each open sentence graphically.

A

1. $|w| = 4$
2. $|x - 3| = 2$
3. $|t| < 3$
4. $|y| > 4$
5. $|u| \geq 2$
6. $|p| \leq 5$
7. $|y - 2| < 3$
8. $|k - 4| > 1$
9. $\left|t - \frac{3}{2}\right| \leq \frac{5}{2}$
10. $\left|x - \frac{1}{2}\right| > \frac{3}{2}$
11. $|r + 2| > 5$
12. $|w + 2| \leq 2$

Solve each open sentence. Use whichever method you prefer.

B **13.** $|2x - 1| = 3$ **14.** $\left|\frac{1}{4}y + 1\right| = \frac{1}{2}$ **15.** $|2p + 5| \geq 3$

16. $|3k - 2| < 4$ **17.** $\left|\frac{1}{3}t - 1\right| \leq \frac{2}{3}$ **18.** $\left|\frac{y}{4} + \frac{1}{2}\right| > \frac{3}{4}$

19. $3 - |2x - 3| > 1$ **20.** $7 - |3y - 2| \leq 1$ **21.** $|9 + 3f| < 4$

22. $|4 + 2y| \geq 3$ **23.** $|1.2 + 0.4t| < 2$ **24.** $|1 - 0.3x| \geq 1.5$

Solve for *x* in terms of the other variables. Assume that *a*, *b*, and *c* are positive numbers.

C **25.** $|x - a| \geq c$ **26.** $|x + a| \leq c$ **27.** $|x - c| < c$

28. $|x + c| > c$ **29.** $|a + bx| \geq c$ **30.** $|a - bx| \leq c$

31. $a - |bx| < c$ (Assume $a - c \geq 0$.) **32.** $a + |bx| > c$ (Assume $c - a \geq 0$.) **33.** $b|x + a| < c$

Mixed Review Exercises

Solve each open sentence.

1. $1 \leq 3x + 4 \leq 13$ **2.** $|5 - 2n| = 3$ **3.** $4(2c - 3) > 7c - 9$

4. $|6y + 6| > 0$ **5.** $|w| + 4 \geq 6$ **6.** $p + 2 < -1$ or $-4p \leq -8$

7. $5z + 11 \leq 1$ **8.** $|t + 4| = 0$ **9.** $\frac{1}{2}m > -1$ and $5 - m > 1$

Self-Test 2

1. Express the following conjunction as an equivalent open sentence involving absolute value: $3x + 2 \geq -4$ and $3x + 2 \leq 4$. Obj. 2-4, p. 73

Solve and graph each solution set.

2. $|2x + 1| > 3$ **3.** $5 - |3 - y| \geq 4$

4. Translate the following statement into an open sentence using absolute value and the variable *x*: The numbers whose distance from -2 is at least 5 units. Obj. 2-5, p. 76

Solve each open sentence graphically.

5. $|m - 4| \leq 2$ **6.** $\left|n + \frac{3}{2}\right| > \frac{1}{2}$

Check your answers with those at the back of the book.

Computer Key-In

In Lesson 2-4 you solved inequalities involving absolute value. The program below solves inequalities of the form $|ax + b| < c$ or $|ax + b| > c$ when the user enters the values of a, b, and c, and the inequality sign. The program uses the following four basic solutions.

For $|ax + b| < c$, where $c > 0$:

If $a > 0$, $\frac{-c-b}{a} < x$ and $x < \frac{c-b}{a}$.   If $a < 0$, $\frac{c-b}{a} < x$ and $x < \frac{-c-b}{a}$.

For $|ax + b| > c$, where $c \geq 0$:

If $a > 0$, $x < \frac{-c-b}{a}$ or $x > \frac{c-b}{a}$.   If $a < 0$, $x < \frac{c-b}{a}$ or $x > \frac{-c-b}{a}$.

The program also handles two special cases, which you will be asked about in Exercises 7 and 8 below.

```
10 PRINT "THIS PROGRAM WILL SOLVE AN INEQUALITY"
20 PRINT "OF THE FORM ABS (A * X + B) SIGN C,"
30 PRINT "WITH A NOT ZERO. THE SIGN IS < OR >."
40 INPUT "ENTER A,B,C, AND THE SIGN: ";A,B,C,S$
50 IF S$ = "<" AND C < = 0 THEN 60
55 PRINT "SOLUTION SET CONSISTS OF X SUCH THAT"
60 IF S$ = "<" AND A > 0 AND C > 0 THEN PRINT
   (-C - B) / A;" < X AND X < "; (C - B) / A
70 IF S$ = "<" AND A < 0 AND C > 0 THEN PRINT
   (C - B) / A;" < X AND X < "; (-C - B) / A
80 IF S$ = ">" AND A > 0 AND C > = 0 THEN PRINT
   "X < ";(-C - B) / A;" OR X > ";(C - B) / A
90 IF S$ = ">" AND A < 0 AND C > = 0 THEN PRINT
   "X < "; (C - B) / A;" OR X > ";(-C - B) / A
100 IF S$ = ">" AND C < 0 THEN PRINT
    "X IS ANY REAL NUMBER."
110 IF S$ = "<" AND C < = 0 THEN PRINT
    "THERE IS NO SOLUTION."
120 END
```

Exercises

Run the program to solve each inequality. Be sure each inequality is in one of the forms $|ax + b| < c$ or $|ax + b| > c$.

1. $|-4x + 5| < 11$
2. $|2x + 9| > 1$
3. $|5x - 1| + 4 > 6$
4. $3|5 - x| > 12$
5. $|2x + 3| - 6 < 7$
6. $-4|x - 1| > -8$

Run the program to solve each inequality. Explain the result.

7. $|3x + 4| < -1$
8. $|-2x + 5| > -3$

Proving Theorems

2-6 Theorems and Proofs

Objective To use axioms, definitions, and theorems to prove some properties of real numbers.

You have used many properties of real numbers in the previous lessons of this book. It can be shown that if just a few of these properties are accepted as true statements, then all the other properties will necessarily follow.

Statements that we assume to be true are called **axioms** (or **postulates**). The axioms that we accept include:

The substitution principle (page 8)
The properties of equality (page 14)
The field properties of real numbers (page 15)
The properties of order (page 59)

At this time, you should review all the properties and the definitions included up to this point.

Example 1 shows how you can reason from a **hypothesis** (a statement that is given or assumed to be true) to a **conclusion** (a statement that follows logically from the assumptions). In the example, each step of reasoning from the hypothesis "a, b, and c are real numbers and $a + c = b + c$" to the conclusion "$a = b$" is justified by a given fact or an axiom.

Example 1 Show that for all real numbers a, b, and c, if $a + c = b + c$, then $a = b$.

Solution

Statements	Reasons
1. a, b, and c are real numbers; $a + c = b + c$.	1. Hypothesis (or Given)
2. $-c$ is a real number.	2. Property of opposites
3. $(a + c) + (-c) = (b + c) + (-c)$	3. Addition property of equality
4. $a + [c + (-c)] = b + [c + (-c)]$	4. Associative property of addition
5. $a + 0 = b + 0$	5. Property of opposites
6. $\therefore a = b$	6. Identity property of addition

This form of logical reasoning from hypothesis to conclusion is called a **proof.** A statement that can be proved is called a **theorem.**

A theorem that can be proved easily from another is called a **corollary.** You can use the commutative property of addition to prove the following corollary of the theorem proved in Example 1.

Corollary: If $c + a = c + b$, then $a = b$.

The theorem of Example 1 and its corollary make up the *cancellation property of addition.*

Cancellation Property of Addition

For all real numbers a, b, and c:

If $a + c = b + c$, then $a = b$.
If $c + a = c + b$, then $a = b$.

A proof of the following property is outlined in Exercise 14.

Cancellation Property of Multiplication

For all real numbers a and b and *nonzero* real numbers c:

If $ac = bc$, then $a = b$.
If $ca = cb$, then $a = b$.

If you interchange the hypothesis and conclusion of an if-then statement, you get the **converse** of the statement. The converse of a true statement is not necessarily true. For example, the converse of "If $a = 1$, then $a^2 = a$" is "If $a^2 = a$, then $a = 1$." This converse is false, as shown by the *counterexample* $0^2 = 0$, but $0 \neq 1$. (In algebra, a **counterexample** is a single numerical example that makes a statement false.)

After a theorem has been proved, it may be used along with axioms and definitions in other proofs. For example, the cancellation property of addition is used in the following proof.

Example 2 Prove the *multiplicative property of 0* (page 27):
For every real number a, $a \cdot 0 = 0$ and $0 \cdot a = 0$.

Proof

Statements	Reasons
1. a is a real number.	1. Given
2. $0 + 0 = 0$	2. Identity property of addition
3. $a \cdot (0 + 0) = a \cdot 0$	3. Multiplication property of equality
4. $a \cdot 0 + a \cdot 0 = a \cdot 0$	4. Distributive property
5. $a \cdot 0 + a \cdot 0 = 0 + a \cdot 0$	5. Identity property of addition
6. $\therefore a \cdot 0 = 0$	6. Cancellation property of addition
7. and $0 \cdot a = 0$	7. Commutative property of multiplication

The multiplicative property of 0 can be used to prove an important property, shown at the top of the next page.

Zero-Product Property

For all real numbers a and b:

$$ab = 0 \text{ if and only if } a = 0 \text{ or } b = 0.$$

An "if and only if" statement such as the one above is equivalent to two "if-then" statements that are converses of each other:

If $ab = 0$, then $a = 0$ or $b = 0$.

If $a = 0$ or $b = 0$, then $ab = 0$.

The second statement follows directly as a corollary to the multiplicative property of 0. A proof of the first statement is outlined in Exercise 11.

The property of opposites (page 15) states that the opposite of a real number *exists* and that it is *unique* (exactly one exists). These facts are used in the two following proofs.

Example 3 Prove: For every real number a, $-(-a) = a$.

Proof

Statements	Reasons
1. a is a real number.	1. Given
2. $-a + a = 0$	2. Property of opposites (existence of $-a$)
3. $-a + [-(-a)] = 0$	3. Property of opposites (existence of $-(-a)$)
4. $\therefore -(-a) = a$	4. Steps 2 and 3 and property of opposites (uniqueness)

Example 4 Prove the property of the *opposite of a product* (page 28): For all real numbers a and b, $-ab = (-a)b$ and $-ab = a(-b)$.

Proof

Statements	Reasons
1. a and b are real numbers.	1. Given
2. $ab + (-a)b = [a + (-a)]b$	2. Distributive property
3. $ab + (-a)b = 0 \cdot b$	3. Property of opposites
4. $ab + (-a)b = 0$	4. Multiplicative property of 0
5. $ab + (-ab) = 0$	5. Property of opposites (existence of $-ab$)
6. $\therefore -ab = (-a)b$	6. Steps 4 and 5 and property of opposites (uniqueness)

(The proof of the second part is outlined in Exercise 5.)

A corollary to the theorem of Example 4 is given in the next example.

Example 5 Prove the *multiplicative property of −1* (page 27):
For every real number a, $a(-1) = -a$ and $(-1)a = -a$.

Proof

Statements	Reasons
1. a is a real number.	1. Given
2. $-a$ is a real number.	2. Property of opposites
3. $-a = -(a \cdot 1)$	3. Identity property of multiplication
4. $-a = a(-1)$	4. Property of the opposite of a product
5. $\therefore a(-1) = -a$	5. Symmetric property of equality
6. $a(-1) = (-1)a$	6. Commutative property of multiplication
7. $\therefore (-1)a = -a$	7. Substitution

Example 6 Prove the property of the *opposite of a sum* (page 29):
For all real numbers a and b, $-(a + b) = (-a) + (-b)$.

Proof

Statements	Reasons
1. a and b are real numbers.	1. Given
2. $-(a + b) = (-1)(a + b)$	2. Multiplicative property of -1
3. $-(a + b) = (-1)a + (-1)b$	3. Distributive property
4. $\therefore -(a + b) = (-a) + (-b)$	4. Multiplicative property of -1

Oral Exercises

For each statement, (a) identify the hypothesis and the conclusion, (b) give the converse of the statement, and (c) tell if the converse is true or false, and if false, give a counterexample.

1. If $x = 1$, then $|x| = 1$.
2. If $|x - 1| = 0$, then $x = 1$.
3. An integer is even if it is divisible by 2.
4. $y^2 > 0$ if $y > 0$.
5. If $0 < x < 1$, then $|x| < 1$.
6. If a triangle is isosceles, then two of its angles are congruent.

Give reasons for the steps shown in each proof. The domain of each variable is the set of real numbers unless otherwise stated.

7. If $a + a = a$, then $a = 0$.

Proof

1. $a + a = a$
2. $-a$ is a real number.
3. $(a + a) + (-a) = a + (-a)$
4. $a + [a + (-a)] = a + (-a)$
5. $a + 0 = 0$
6. $\therefore a = 0$

8. If $b \neq 0$, then $\frac{ab}{b} = a$.

Proof

1. $b \neq 0$
2. $\frac{1}{b}$ is a real number.
3. $\frac{ab}{b} = (ab) \cdot \frac{1}{b}$
4. $\frac{ab}{b} = a\left(b \cdot \frac{1}{b}\right)$
5. $\frac{ab}{b} = a \cdot 1$
6. $\therefore \frac{ab}{b} = a$

Written Exercises

Give a counterexample to show that each statement is false. The domain of each variable is the set of real numbers.

A **1.** If $a^2 = b^2$, then $a = b$.

2. If $b < a$, then $a - b < 0$.

3. $|a - b| = |a| - |b|$

4. $|-a| < 0$

Give reasons for the steps shown in each proof. You may use the axioms listed at the top of page 81, definitions, theorems proved in the examples of this lesson, and the results of earlier exercises as reasons. The domain of each variable is the set of real numbers unless otherwise stated.

5. The opposite of a product: $-ab = a(-b)$.

Proof

1. $-ab = -ba$
2. $-ab = (-b)a$ (See Example 4.)
3. $\therefore -ab = a(-b)$

6. The product of opposites: $(-a)(-b) = ab$.

Proof

1. $(-a)(-b) = -[a(-b)]$
2. $(-a)(-b) = -[-(ab)]$
3. $\therefore (-a)(-b) = ab$ (See Example 3.)

7. $(a + b) - b = a$.

Proof

1. $(a + b) - b = (a + b) + (-b)$
2. $(a + b) - b = a + [b + (-b)]$
3. $(a + b) - b = a + 0$
4. $\therefore (a + b) - b = a$

8. If $x + c = 0$, then $x = -c$.

Proof

1. $x + c = 0$
2. $(x + c) + (-c) = 0 + (-c)$
3. $x + [c + (-c)] = 0 + (-c)$
4. $x + 0 = 0 + (-c)$
5. $\therefore x = -c$

9. If $u \neq 0$ and $u^2 = u$, then $u = 1$.

Proof

1. $u \neq 0$; $u^2 = u$
2. $u \cdot u = u$
3. $\frac{1}{u}$ is a real number.
4. $(u \cdot u) \cdot \frac{1}{u} = u \cdot \frac{1}{u}$
5. $u \cdot \left(u \cdot \frac{1}{u}\right) = u \cdot \frac{1}{u}$
6. $u \cdot 1 = 1$
7. $\therefore u = 1$

10. If $c \neq 0$ and $cx = 1$, then $x = \frac{1}{c}$.

Proof

1. $c \neq 0$; $cx = 1$
2. $\frac{1}{c}$ is a real number.
3. $\frac{1}{c}(cx) = \frac{1}{c} \cdot 1$
4. $\left(\frac{1}{c} \cdot c\right)x = \frac{1}{c} \cdot 1$
5. $1 \cdot x = \frac{1}{c} \cdot 1$
6. $\therefore x = \frac{1}{c}$

11. Zero-product property: If $ab = 0$, then $a = 0$ or $b = 0$.

Proof

1. $ab = 0$
2. Suppose $a \neq 0$. Then $\frac{1}{a}$ is a real number. (Note that if $a = 0$, the given statement is obviously true.)
3. $\frac{1}{a}(ab) = \frac{1}{a} \cdot 0$
4. $\frac{1}{a}(ab) = 0$
5. $\left(\frac{1}{a} \cdot a\right)b = 0$
6. $1 \cdot b = 0$
7. $b = 0$
8. $\therefore$ if $a \neq 0$, then $b = 0$.

12. Multiplication is distributive with respect to subtraction: $a(b - c) = ab - ac$.

Proof

1. $a(b - c) = a[b + (-c)]$
2. $a(b - c) = ab + a(-c)$
3. $a(b - c) = ab + (-ac)$ (*Hint:* Use Exercise 5.)
4. $\therefore a(b - c) = ab - ac$

13. Division is distributive with respect to subtraction: If $c \neq 0$, then $\frac{a - b}{c} = \frac{a}{c} - \frac{b}{c}$.

Proof

1. $c \neq 0$; $\frac{1}{c}$ is a real number.
2. $\frac{a - b}{c} = (a - b) \cdot \frac{1}{c}$
3. $\frac{a - b}{c} = a \cdot \frac{1}{c} - b \cdot \frac{1}{c}$ (*Hint:* Use Exercise 12.)
4. $\therefore \frac{a - b}{c} = \frac{a}{c} - \frac{b}{c}$

14. Cancellation property of multiplication: If $c \neq 0$ and $ca = cb$, then $a = b$.

Proof

1. $c \neq 0;\ ca = cb$
2. $\frac{1}{c}$ is a real number.
3. $\frac{1}{c}(ca) = \frac{1}{c}(cb)$
4. $\left(\frac{1}{c} \cdot c\right)a = \left(\frac{1}{c} \cdot c\right)b$
5. $1 \cdot a = 1 \cdot b$
6. $\therefore a = b$

B **15.** Follow the steps outlined in Example 3 (with multiplication replacing addition) to prove that if $a \neq 0$, then $\frac{1}{\frac{1}{a}} = a$.

16. Prove: If $b \neq 0$, then $\frac{-a}{b} = -\frac{a}{b}$.
(*Hint:* Use the definition of division and Example 4.)

17. Prove: If $b \neq 0$, then $\frac{1}{-b} = -\frac{1}{b}$.
$\left(\textit{Hint:}\right.$ Show that $-\frac{1}{b}$ is the reciprocal of $-b$.$\left.\right)$

18. Prove: If $b \neq 0$, then $\frac{a}{-b} = -\frac{a}{b}$. (*Hint:* Use Exercises 17 and 5.)

19. Prove: If $b \neq 0$, then $\frac{-a}{-b} = \frac{a}{b}$. (*Hint:* Use Exercises 17 and 6.)

20. Prove: If $a \neq 0$ and $b \neq 0$, then $\frac{1}{a} \cdot \frac{1}{b} = \frac{1}{ab}$.
(*Hint:* Show that $\frac{1}{a} \cdot \frac{1}{b}$ is the reciprocal of ab.)

21. Prove: If $b \neq 0$ and $d \neq 0$, then $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$.
(*Hint:* Use Exercise 20.)

C **22.** Prove: If $c \neq 0$ and $d \neq 0$, then $\frac{1}{\frac{c}{d}} = \frac{d}{c}$.

23. Prove: If $c \neq 0$ and $d \neq 0$, then $\frac{a}{b} \div \frac{c}{d} = \frac{ad}{bc}$.

24. Use the zero-product property to give a convincing argument why 0 has no reciprocal.

25. Use the zero-product property to give a convincing argument why the product of two nonzero real numbers is never equal to 0.

2-7 Theorems about Order and Absolute Value

Objective To prove theorems about inequalities and absolute value.

The properties of order (page 59) play an important part in proving the theorems of this section.

Example 1 Prove: If a is a real number and $a \neq 0$, then $a^2 > 0$.

Proof

Statements	Reasons
1. a is a real number; $a \neq 0$	1. Given
2. Either $a > 0$ or $a < 0$.	2. Comparison property of order
3. *Case 1:* $a > 0$ Multiply this inequality by the *positive* number a: $\underbrace{a \cdot a} > \underbrace{a \cdot 0}$	3. First multiplication property of order (with $<$ replaced by $>$)
4. $\therefore\ a^2 > 0$	4. Definition of a^2 (page 7) and multiplicative property of 0
5. *Case 2:* $a < 0$ Multiply this inequality by the *negative* number a: $\underbrace{a \cdot a} > \underbrace{a \cdot 0}$	5. Second multiplication property of order
6. $\therefore\ a^2 > 0$	6. See the reasons in Step 4.

The rules for multiplication are proved as theorems in Examples 2 and 3.

Example 2 Prove for all real numbers a and b:

a. If $a > 0$ and $b > 0$, then $ab > 0$.
b. If $a < 0$ and $b < 0$, then $ab > 0$.
c. If $a > 0$ and $b < 0$, then $ab < 0$.

Proof We will prove parts (**b**) and (**c**). The proof of (**a**) is left as Exercise 9.

*Proof of part (**b**)*

Statements	Reasons
1. $a < 0$ and $b < 0$	1. Given
2. Multiply both sides of $b < 0$ by the *negative* number a: $ab > a \cdot 0$	2. Second multiplication property of order
3. $\therefore\ ab > 0$	3. Multiplicative property of 0

Proof of part (c)

Statements	Reasons
1. $a > 0$ and $b < 0$	1. Given
2. Multiply both sides of $b < 0$ by the *positive* number a: $ab < a \cdot 0$	2. First multiplication property of order
3. $\therefore ab < 0$	3. Multiplicative property of 0

Recall that the **absolute value** of a real number x is defined as follows:

$$|x| = x \quad \text{if } x \geq 0$$
$$|x| = -x \text{ if } x < 0$$

Example 3 Prove: For all real numbers a and b, $|ab| = |a| \cdot |b|$.

Proof If either a or b is 0, $|ab| = |a| \cdot |b|$ because both products are 0. There are three remaining cases to prove. Case 1: $a > 0$ and $b > 0$; Case 2: $a < 0$ and $b < 0$; and Case 3: $a > 0$ and $b < 0$. Cases 2 and 3 follow. Case 1 is left as Exercise 10.

Case 2: $a < 0$ and $b < 0$

Statements	Reasons
1. $a < 0$; $b < 0$	1. Given
2. $ab > 0$	2. Example 2, part (b)
3. $\|ab\| = ab$	3. Definition of absolute value
4. $\|a\| = -a$, $\|b\| = -b$	4. Definition of absolute value
5. $\|a\| \cdot \|b\| = (-a)(-b)$	5. Multiplication property of equality
6. $\|a\| \cdot \|b\| = ab$	6. Exercise 6, page 85
7. $\therefore \|ab\| = \|a\| \cdot \|b\|$	7. Substitution principle (Steps 3 and 6)

Case 3: $a > 0$ and $b < 0$

Statements	Reasons
1. $a > 0$; $b < 0$	1. Given
2. $ab < 0$	2. Example 2, part (c)
3. $\|ab\| = -ab$	3. Definition of absolute value
4. $\|a\| = a$, $\|b\| = -b$	4. Definition of absolute value
5. $\|a\| \cdot \|b\| = a(-b)$	5. Multiplication property of equality
6. $\|a\| \cdot \|b\| = -ab$	6. Property of the opposite of a product
7. $\therefore \|ab\| = \|a\| \cdot \|b\|$	7. Substitution principle (Steps 3 and 6)

Oral Exercises

Give reasons for the steps shown in each proof.

1. Prove: For all real numbers a and b, if $a < b$, then $-a > -b$.

Proof

1. $a < b$
2. $(-1)a > (-1)b$
3. $\therefore -a > -b$

2. Prove: If a is a real number and $a < 0$, then $-a > 0$.

Proof

1. $a < 0$
2. $a + (-a) < 0 + (-a)$
3. $0 < 0 + (-a)$
4. $\therefore 0 < -a$, or $-a > 0$

3. In Example 3, why is there no need to consider the case $a < 0$ and $b > 0$?

Written Exercises

Give reasons for the steps shown in the proof of each theorem. The domain of each variable is the set of real numbers unless otherwise stated.

A **1.** If $a > b$, then $-a < -b$.

Proof

1. $a > b$
2. $(-1)a < (-1)b$
3. $\therefore -a < -b$

2. If $a > b$, then $a - c > b - c$.

Proof

1. $a > b$
2. $a + (-c) > b + (-c)$
3. $\therefore a - c > b - c$

3. If $a < b$ and $c < d$, then $a + c < b + d$.

Proof

1. $a < b$
2. $a + c < b + c$
3. $c < d$
4. $b + c < b + d$
5. $\therefore a + c < b + d$

4. If $0 < a < b$, then $a^2 < b^2$.

Proof

1. $0 < a < b$
2. $a \cdot a < a \cdot b$
3. $a \cdot b < b \cdot b$
4. $a \cdot a < b \cdot b$
5. $\therefore a^2 < b^2$

5. If $a < b < 0$, then $a^2 > b^2$.

Proof

1. $a < b$ and $a < 0$
2. $a \cdot a > a \cdot b$
3. $a < b$ and $b < 0$
4. $a \cdot b > b \cdot b$
5. $a \cdot a > b \cdot b$
6. $\therefore a^2 > b^2$

6. If $0 < a < b$ and $0 < c < d$, then $ac < bd$.

Proof

1. $a < b$ and $c > 0$
2. $ac < bc$
3. $c < d$ and $b > 0$
4. $bc < bd$
5. $\therefore ac < bd$

7. Prove: If $a > 0$, then $-a < 0$. (*Hint:* See Oral Exercise 2.)
8. Prove: If $0 < a < 1$, then $a^2 < a$.
9. Prove: If $a > 0$ and $b > 0$, then $ab > 0$. (Part (a), Example 2)
10. Prove: If $a > 0$ and $b > 0$, then $|ab| = |a|\,|b|$. (Case 1, Example 3)

In Exercises 11–16, assume that if $a > 0$, then $\frac{1}{a} > 0$, and if $a < 0$, then $\frac{1}{a} < 0$.

B 11. Prove: If $a > 0$ and $b > 0$, then $\frac{a}{b} > 0$.

12. Prove: If $a < 0$ and $b < 0$, then $\frac{a}{b} > 0$.

13. Prove: If $a > 0$ and $b < 0$, then $\frac{a}{b} < 0$.

14. Prove: If $a < 0$ and $b > 0$, then $\frac{a}{b} < 0$.

15. Prove: If $a > 0$, $b > 0$, and $\frac{1}{a} > \frac{1}{b}$, then $a < b$.

16. Prove: If $0 < a < b$, then $\frac{1}{a} > \frac{1}{b}$. $\left(\textit{Hint:} \text{ Multiply by } \frac{1}{ab}.\right)$

C 17. Prove: If $a < b$, then $a < \frac{a+b}{2} < b$. This proof shows that the average of two numbers lies between them.

18. Prove: For every $a > 0$, if $a > \frac{1}{a}$, then $a > 1$. $\Big($*Hint:* Give an *indirect proof.* That is, show that the assumptions $a = 1$ and $a < 1$ each lead to a contradiction of the hypothesis that $a > \frac{1}{a}$.$\Big)$

Mixed Review Exercises

Tell whether each statement is true for all real numbers.

1. If $a > b$, then $b < a$.
2. If $a > b$ and $c < 0$, then $ac < bc$.
3. If $ac = bc$ and $c \neq 0$, then $a = b$.

Solve each open sentence and graph each solution set that is not empty.

4. $|x - 3| = 1$
5. $4d + 5 \geq 1$
6. $-1 < 2 - y < 3$
7. $6r + 13 = 25$
8. $|n| + 7 < 5$
9. $-2k > 8$ or $k - 4 \geq 0$

Self-Test 3

Vocabulary

axiom (p. 81)
hypothesis (p. 81)
conclusion (p. 81)
proof (p. 81)
theorem (p. 81)
corollary (p. 81)
converse (p. 82)
counterexample (p. 82)

1. Give reasons for the steps shown in the following proof. If $a + b = c$, then $a = c - b$. **Obj. 2-6, p. 81**

Proof

1. $a + b = c$
2. $-b$ is a real number.
3. $(a + b) + (-b) = c + (-b)$
4. $a + [b + (-b)] = c + (-b)$
5. $a + 0 = c + (-b)$
6. $a = c + (-b)$
7. $\therefore a = c - b$

2. Prove: If $ab = c$ and $b \neq 0$, then $a = \frac{c}{b}$.

3. Prove: If $a > 1$, then $a^2 > a$. **Obj. 2-7, p. 88**

Chapter Summary

1. Inequalities can be solved by using the properties of order on page 59 and the transformations on page 60.

2. If an inequality is a conjunction, it can be solved by finding the values of the variable for which both sentences are true. If an inequality is a disjunction, it can be solved by finding the values of the variable for which at least one of the sentences is true.

3. Some word problems can be solved algebraically by translating the given information into an inequality and then solving the inequality.

4. Equations and inequalities that involve absolute value can be solved algebraically by writing an equivalent conjunction or disjunction.

5. Equations and inequalities that involve absolute value can be solved geometrically by using this fact: On a number line the distance between the graphs of two numbers is the absolute value of the difference between the numbers.

6. Axioms are statements assumed to be true. Using the axioms for real numbers (referred to on page 81), other properties of real numbers can be proved as theorems. Every step in the proof of a theorem can be justified by either an axiom, a definition, a given fact, or a theorem.

Chapter Review

Give the letter of the correct answer.

Solve.

1. $-\frac{m}{2} < -2$ **2-1**

a. $m > 1$ **b.** $m > 4$ **c.** $m < 4$ **d.** $m > -4$

2. $3(n - 1) > 5n + 7$

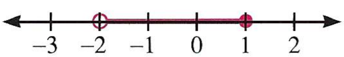
a. $n > -5$ **b.** $n < -4$ **c.** $n > -4$ **d.** $n < -5$

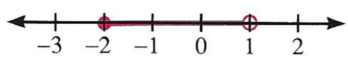
3. $-3 < 4c + 5 \le 1$ **2-2**

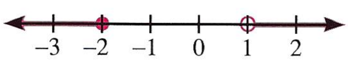
a. $-2 < c \le -1$ **b.** $-2 < c \le 1$

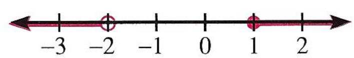
c. $2 < c \le -1$ **d.** $2 < c \le 1$

4. Graph the solution set of the following disjunction:
$4 - w \le 3$ or $w + 5 < 3$

a. 

b. 

c. 

d. 

Solve.

5. At the video store, you can rent tapes for $1.50 each per day, or you can get unlimited rentals for a monthly fee of $20. At most how many tapes could you rent at $1.50 before spending more than the cost for a month's unlimited rentals? **2-3**

a. 12 **b.** 13 **c.** 14 **d.** 15

6. $\left|5 - \frac{x}{3}\right| = 7$ **2-4**

a. $\{-6, 36\}$ **b.** $\{-6, -36\}$ **c.** $\{6, -36\}$ **d.** $\{6, 36\}$

7. $|2y + 9| < 13$

a. $-2 < y < 11$ **b.** $y > 2$ **c.** $-11 < y < 2$ **d.** $y < -11$ or $y > 2$

8. $|4 - h| \ge 5$

a. $h \le -1$ or $h \ge 9$ **b.** $h \le -1$

c. $h \le -9$ or $h \ge 1$ **d.** $-1 \le h \le 9$

9. Describe the graph of the solution set of $|k + 3| < 2$. **2-5**

a. Points at least 2 units from 3. **b.** Points less than 2 units from -3.

c. Points more than 3 units from 2. **d.** Points at most 2 units from -3.

10. A theorem that can be proved easily from another theorem is a(n) __?__. **2-6**

a. axiom **b.** corollary **c.** property **d.** theorem

11. The converse of a true statement is __?__.

a. a corollary **b.** never true **c.** always true **d.** sometimes true

Supply the conclusion that makes each statement true.

12. For real numbers a and b, $ab = 0$ if and only if _?_.

a. $a = 0$ **b.** $b = 0$ **c.** $a = 0$ or $b = 0$ **d.** $a = 0$ and $b = 0$

13. For real numbers a, b, and c, if $a > b$ and $c < 0$, then _?_. 2-7

a. $\frac{a}{c} > \frac{b}{c}$ **b.** $ac > bc$ **c.** $\frac{a}{c} < \frac{b}{c}$ **d.** $a + c < b + c$

Chapter Test

Solve.

1. $5x - 9 > 6x$ **2.** $3(2y + 1) < 2(y - 3) + 1$ 2-1

3. $-3 < 4 - m < 6$ **4.** $-2n \geq 8$ or $n + 3 < 7$ 2-2

5. The dimensions of a rectangle are consecutive odd integers. Find the smallest such rectangle with a perimeter of at least 35 cm. 2-3

Solve.

6. $|9 - 2k| = 5$ **7.** $\left|\frac{c}{3} + 1\right| > 2$ **8.** $|4f + 3| \leq 5$ 2-4

Solve each sentence graphically.

9. $|w - 4| \geq 1$ **10.** $|h + 2| < 3$ 2-5

11. Write the converse of the following statement and tell whether the converse is true or false: "If $x > 1$, then $x^2 > x$." 2-6

12. Supply reasons for the steps in the proof of the following statement: For all real numbers a and nonzero real numbers b, if $ab = b$, then $a = 1$.

1. a is a real number; b is a nonzero real number; $ab = b$	1. _?_
2. $\frac{1}{b}$ is a real number.	2. _?_
3. $(ab) \cdot \frac{1}{b} = b \cdot \frac{1}{b}$	3. _?_
4. $a\left(b \cdot \frac{1}{b}\right) = b \cdot \frac{1}{b}$	4. _?_
5. $a(1) = 1$	5. _?_
6. $a = 1$	6. _?_

Tell whether the statement is true for all real numbers. If it is not, give a counterexample.

13. If $a > b$, then $|a| > |b|$. **14.** If $a < b$ and $c > d$, then $ad < bc$. 2-7

15. If $a < b$ and $c < 0$, then $ac < bc$. **16.** If $a < b < 0$, then $a^2 < b^2$.

Extra / Symbolic Logic: Boolean Algebra

Most people think of algebra as the study of operations with numbers and variables. Another kind of algebra that is used in the design of electronic digital computers involves operations with logical statements. In fact, complex electrical circuits can be analyzed using logical statements similar to those that you will work with in Exercises 1–18. This "algebra of logic" is called **Boolean algebra** in honor of its originator, George Boole (British, 1815–1864).

In Boolean algebra you use letters such as p, q, r, s, and so on, to stand for statements. For example, you might let p represent the statement "3 is an odd integer," and q, the statement "5 is less than 3." In this case, the statement p has the truth value T (the statement is true), whereas q has the truth value F (the statement is false).

The table below shows the operations that are used in Boolean algebra to produce compound statements from any given statements p and q.

Operation	In words	In symbols
Conjunction	p and q	$p \wedge q$
Disjunction	p or q	$p \vee q$
Conditional	If p, then q	$p \rightarrow q$
Equivalence	p if and only if q	$p \leftrightarrow q$
Negation	not p	$\sim p$

Note that in ordinary English the word *or* is sometimes used in the exclusive sense, to mean that just one of two alternatives occurs. In mathematical usage, however, the word *or* generally is used in the inclusive sense. As you can see in the table below, the disjunction p or q is true if only p is true, if only q is true, or if both are true. The rules for assigning truth values to compound statements are shown in the following five *truth tables*.

The conjunction $p \wedge q$ is true whenever *both* p and q are true.

Conjunction

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

The disjunction $p \vee q$ is true whenever *at least one* of the statements is true.

Disjunction

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

The conditional $p \rightarrow q$ is false only when p is true and q is false.

Conditional

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

The equivalence $p \leftrightarrow q$ is true whenever p and q have the same truth value.

Equivalence

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

The negation $\sim p$ is the denial of p. Therefore, it is reasonable to agree that $\sim p$ is false when p is true, and true when p is false.

Negation

p	$\sim p$
T	F
F	T

Example 1 Let r stand for "$1 > 2$," and s for "$2 < 4$." Read each of the following statements. Then by referring to the truth tables above and on page 95, give the truth value of the statement and a reason for your answer.

a. $r \wedge s$ **b.** $r \vee s$ **c.** $r \rightarrow s$ **d.** $r \leftrightarrow s$ **e.** $\sim r$

Solution

a. $r \wedge s$: $1 > 2$ and $2 < 4$. F; the truth value of r is F.

b. $r \vee s$: $1 > 2$ or $2 < 4$. T; the truth value of s is T.

c. $r \rightarrow s$: If $1 > 2$, then $2 < 4$. T; the truth value of r is F, and that of s is T.

d. $r \leftrightarrow s$: $1 > 2$ if and only if $2 < 4$. F; r and s have different truth values.

e. $\sim r$: not $(1 > 2)$, that is $1 \leq 2$. T; the truth value of r is F.

Two compound statements are *logically equivalent* if they have the same truth value for each combination of truth values of the individual statements.

Example 2 Construct a truth table for $q \vee \sim p$ and $p \rightarrow q$ for all possible truth values of p and q. Show that $q \vee \sim p$ and $p \rightarrow q$ are logically equivalent.

Solution

p	q	$\sim p$	$q \vee \sim p$	$p \rightarrow q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Column 4 was constructed by using disjunction with columns 2 and 3.

Column 5 was constructed by using the conditional with columns 1 and 2.

Since columns 4 and 5 match, $q \vee \sim p$ and $p \rightarrow q$ are logically equivalent.

A compound statement that is true for all truth values of its component statements is called a **tautology.** For example, $p \leftrightarrow p$ is a tautology.

Exercises

In Exercises 1–9 assume that r and p are true statements and that q is a false statement. Determine the truth value of each statement.

1. $q \rightarrow r$
2. $\sim r \wedge p$
3. $p \vee \sim q$
4. $r \wedge (p \vee q)$
5. $p \vee (q \wedge r)$
6. $(p \vee r) \rightarrow q$
7. $\sim p \rightarrow (q \vee \sim r)$
8. $r \rightarrow (q \rightarrow r)$
9. $[r \wedge (p \vee q)] \rightarrow [(r \wedge p) \vee (r \wedge q)]$

Given that $p \rightarrow q$ is false, show that each statement in Exercises 10–12 is true.

10. $q \rightarrow p$
11. $p \wedge \sim q$
12. $(p \vee q) \wedge p$

Construct a truth table and determine if the two statements in each exercise are logically equivalent.

13. $p \vee q$, $q \vee p$
14. $q \vee \sim q$, q

Construct a truth table for each statement and determine whether the statement is a tautology.

15. $(p \vee q) \rightarrow p$
16. $(q \wedge \sim q) \rightarrow p$
17. $r \wedge (p \vee q) \leftrightarrow (r \wedge p) \vee (r \wedge q)$
18. $[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$

Mixed Problem Solving

Solve each problem that has a solution. If a problem has no solution, explain why.

A **1.** A child has 40 coins worth more than \$6.00. If the coins are dimes and quarters only, what can you conclude about the number of quarters?

2. The width and length of a rectangle are consecutive odd integers. If the width is w, find the perimeter, in simplest form, in terms of w.

3. Percy works 8 hours more per week than Selena. Selena, however, earns \$2 more per hour than Percy, who earns \$6 per hour. If their weekly pay is the same, how many hours does Percy work per week?

4. In Sioux Falls, South Dakota, the average low temperatures in January and July are $-16°$ C and $17°$ C, respectively. Find the difference between these temperatures.

5. Marcus invested \$4000 at simple annual interest, some at 5% and the rest at 8%. If he received \$272 in interest for one year, how much did he invest at each rate?

6. Megan bought four times as many pencils as erasers. Pencils cost \$.04 each, and erasers cost \$.19 each. If Megan spent \$2.10 in all, how many pencils did she buy?

7. Four tennis balls cost d dollars. How many tennis balls can you buy for \$8? Give your answer in terms of d.

8. Drew's second test score was 6 points higher than his first score. If his average was less than 80, what can you conclude about his first score?

9. The measure of a supplement of an angle is four times the measure of a complement. Find the measure of the angle.

10. Helen has \$57 more in her checking account than in her savings account. If she has \$239 altogether, how much is in savings?

B **11.** At a health food store, dried apple slices that cost \$1.80/lb are mixed with dried banana slices that cost \$2.10/lb. If the mixture weighs 5 lb and costs \$1.92/lb, how many pounds of apple slices were used?

12. The Petersons drove to a friend's house at 80 km/h. On the return trip, rain reduced their speed by 10 km/h. If their total driving time was 3 h, find the total distance driven.

13. The width of a rectangle is 1 cm more than half the length. When both the length and the width are increased by 1 cm, the area increases by 20 cm^2. Find the original dimensions of the rectangle.

14. At 1:00 P.M. Sue left her home and began walking at 6 km/h toward Sandy's house. Fifteen minutes later, Sandy left her home and walked at 8 km/h toward Sue's house. If Sue lives 5 km from Sandy, at what time did they meet? Who walked farther?

Preparing for College Entrance Exams

Strategy for Success

If there is a penalty for incorrect answers, random guessing may not be to your advantage. However, if you can eliminate several answers in a given question and then take a guess, your chance of selecting the correct answer is improved and it may be worthwhile to guess.

Decide which is the best of the choices given and write the corresponding letter on your answer sheet.

1. If $a > b$ and $c < 0$, which of the following *must* be true:

I. $ab > bc$ II. $a^2 > b^2$ III. $\frac{a+c}{c} < \frac{b+c}{c}$

(A) I only **(B)** II only **(C)** III only **(D)** I, II, and III **(E)** II and III only

2. Which of the following open sentences is true for all real numbers x?

(A) $3[x - 5(2 - x)] = 2(9x + 15)$ **(B)** $|7 - x| > 0$

(C) $(7x - 1) \div \frac{1}{2} = 2 - 14x$ **(D)** $-\frac{5}{2}(-6 + 4x) = 10\left(\frac{3}{2} - x\right)$

(E) $x + 3 < 3 - x$

3. Lucy spent \$2.36 on 15¢ and 22¢ stamps. If she bought twice as many 22¢ stamps as 15¢ stamps, how many 22¢ stamps did she buy?

(A) 2 **(B)** 4 **(C)** 8 **(D)** 5 **(E)** 10

4. Evaluate $x^3 - 3x^2 \div (4 + x)$ if $x = -2$.

(A) 2 **(B)** -2 **(C)** -10 **(D)** -14 **(E)** -6

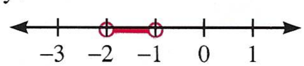
5. Solve $A = \frac{h}{2}(b_1 + b_2)$ for b_2.

(A) $\frac{A}{2h} - b_1$ **(B)** $A - \frac{hb_1}{2}$ **(C)** $\frac{2A - b_1}{h}$ **(D)** $\frac{2A}{h} - b_1$ **(E)** $\frac{2A}{hb_1}$

6. How many sets of four consecutive integers are there such that half the sum of the first three is greater than the fourth, and the sum of the last three is at least four times the first?

(A) none **(B)** one **(C)** two **(D)** three **(E)** four

7. The graph at the right shows the solution set to which inequality?



(A) $|2t + 3| < 1$ **(B)** $|2t + 3| > 1$

(C) $|3t + 2| < 1$ **(D)** $|3t + 2| \geq -1$

(E) $\frac{1}{2} \geq \left|t + \frac{3}{2}\right|$